

Plan

- 1 Key Problems: 5.1e, 5.2de, 5.3cd, 5.4, 5.5b, 6.1, 6.2cd, 6.3abc, 6.4abce, 6.5ab, 7.2b, (7.3b, 7.4cd)
- 2 Midterm Exams: 10/2022 Q6-8

① Key problems

5.1 e) $A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix} > 0 \Rightarrow \text{regular}$

$\lambda = 1$: $\begin{pmatrix} -0.25 & 0.02 & 0.1 \\ 0.20 & -0.90 & 0.2 \\ 0.05 & 0.08 & -0.3 \end{pmatrix} \rightarrow \begin{pmatrix} 0.05 & 0.08 & -0.3 \\ 0 & -0.42 & 1.4 \\ 0 & 0 & 0 \end{pmatrix}$ $0.05x = -0.3z$
 $-0.42y = -1.4z$
 z free

$$y = \frac{-1.4z}{-0.42} = \frac{140}{42} z = \frac{10}{3} z$$

$$0.05x = -0.08 \left(\frac{10}{3} z \right) + 0.3z \quad | \cdot 20$$

$$x = -\frac{16}{3} z + 6z = \frac{2}{3} z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{z}{3} \begin{pmatrix} 2 \\ 10 \\ 3 \end{pmatrix}$$

$$\underline{v} = \frac{1}{15} \begin{pmatrix} 2 \\ 10 \\ 3 \end{pmatrix} \stackrel{15}{=} \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}$$

5.2 d) $A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & 7 & 0 \\ -5 & 0 & 35 \end{pmatrix}$ $D_1 = 2$ $\Delta_1 = 2, 7, 35 \geq 0$
 $D_2 = 5$ $\Delta_2 = 5, 7, 35, 70 - 25 \geq 0$
 $D_3 = -5(35) + 35 \cdot 5 = 0$ $\Delta_3 = 0 \geq 0$

RRC: $\text{rk } A = 2$
A pos. semidefn

$$c) A = \begin{pmatrix} -1 & -2 & -2 \\ -2 & -4 & -4 \\ -2 & -4 & -2 \end{pmatrix}$$

$$R(2) = 2R(1)$$

$$\text{rk } A = 2$$

$$\begin{aligned} D_1 &= -1 \\ D_2 &= 0 \\ D_3 &= 0 \end{aligned}$$

$$\begin{aligned} \Delta_1 &= -1, -4, -2 < 0 \\ \Delta_2 &= 0, \textcircled{-8} \text{ not } \geq 0 \end{aligned}$$

A indefinite

$$5.3 \quad c) f = \frac{3x^2}{2} + \frac{4xy}{2} - \frac{4xz}{2} + \frac{3y^2}{2} + \frac{4yz}{2} + \frac{8z^2}{2}$$

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 8 \end{pmatrix} \quad \begin{aligned} D_1 &= 3 \\ D_2 &= 5 \\ D_3 &= -2 \cdot 10 - 2 \cdot 10 + 8 \cdot 5 = 0 \end{aligned}$$

$$\text{RRC: } \text{rk } A = 2$$

$\Rightarrow$ pos. semidefn.

$$d) f = xw - yz$$

" $f(x, y, z, w)$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} D_1 &= 0 & \Delta_1 &= 0, 0, 0, 0 \\ D_2 &= 0 & \Delta_2 &= 0, -1/4 \text{ not } \geq 0 \\ D_3 &= 0 \\ D_4 &= -\frac{1}{2} \cdot \frac{1}{2} (-1/4) = \frac{1}{16} \end{aligned}$$

$$\Delta_2 = M_{2,2,2,2} = 0 - 1/4 = -1/4 < 0 \quad \text{indefinite}$$

5.4

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} D_1 &= 1 \\ D_2 &= 1 \\ D_3 &= 1 \cdot 0 = 0 \\ D_4 &= 0 \end{aligned}$$

$$R(3) = -R(2)$$

$$R(4) = -R(1)$$

$\left. \begin{array}{l} R(3) = -R(2) \\ R(4) = -R(1) \end{array} \right\} \text{rk } A = 2 : \text{RRC} \Rightarrow \text{pos. semidefn.}$

5.5 b)

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

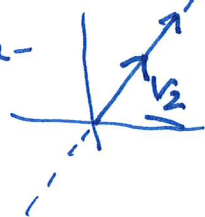
ordinary diagonalization

where each vector has length one and

$$\underline{v}_1 \cdot \underline{v}_2 = \underline{v}_1 \cdot \underline{v}_3 = \underline{v}_2 \cdot \underline{v}_3 = 0$$

since A symmetric, and $\underline{v}_1, \underline{v}_2, \underline{v}_3$ have

distinct eigenvalues.

 $\Rightarrow$ orthogonal diagonalization

Eigenvalues: $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$

$$(4-\lambda) \cdot \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \cdot (\lambda^2 - 8\lambda + 15) = 0$$

$$\underline{\lambda_1 = 4} \quad \underline{\lambda_2 = 3}, \underline{\lambda_3 = 5}$$

Eigenvectors:

$$\underline{\lambda = 4}: \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{array}{l} z=0 \\ y \text{ free} \\ x=0 \end{array}$$

$$\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{\underline{v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}}$$

$$\underline{\lambda = 3}: \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{array}{l} x+z=0 \\ y=0 \\ z \text{ free} \end{array}$$

$$\begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\underline{\underline{v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}}$$

$$\underline{\lambda = 5}: \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{array}{l} -x+z=0 \\ -y=0 \\ z \text{ free} \end{array}$$

$$\begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \underline{\underline{v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}}$$

5.5 c): See very similar example in the start of Lecture 6.

6.1 $f = x^2 + ay^2 - 2axyz + az^2 + w^2 - 2xw$ (a parameter)

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & a & -a & 0 \\ 0 & -a & a & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} D_1 &= 1 \\ D_2 &= a \\ D_3 &= 0 \\ D_4 &= 0 \end{aligned}$$

a) $a > 0$: $\text{rk} A = 2$
RRE: pos. semidefn.

b) $a < 0$: $\text{rk} A = 1$
RRE: pos. semidefn.

c) $a < 0$: $D_2 < 0$ indefn.

$$\begin{aligned} R(3) &= -R(2) \\ R(4) &= -R(1) \end{aligned}$$

6.2 c) $f = 2xy + 2xz + 2yw + 2zw$
 $= \underline{x}^T A \underline{x}$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} D_1 &= 0 \\ D_2 &= -1 < 0 \end{aligned}$$

A indefn.

(no max/min for f)

A pos. semidefn.: $f_{\min} = 0$ global min
A neg. semidefn.: $f_{\max} = 0$ " max
A indefn.: no global max/min

Stat pts: $f'(\underline{x}) = 2A\underline{x} = \underline{0}$
 $A\underline{x} = \underline{0}$
 $\Rightarrow (x, y, z, w) = (-w, -z, z, w)$ with z, w free

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x &= -w \\ y &= -z \\ z, w &\text{ free} \end{aligned}$$

$(0, 0, 0, 0)$: $H(f)(0, 0, 0, 0) = 2A$ indefn. $\Rightarrow$ saddle pt

d) $f = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$
 $= \underline{x}^T A \underline{x}$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

A pos. defn.



$$f(0, 0, 0, 0) = 0$$

$$f_{\min} = 0$$

$$\begin{aligned} D_1 &= 1 \\ D_2 &= 3/4 \\ D_3 &= -1/2 \cdot 1/2 + 1 \cdot 3/4 = 1/4 = 1/2 \\ D_4 &= -1/2 \cdot \begin{vmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/2 & 1 \end{vmatrix} + 1 \cdot 1/2 \\ &= -\frac{1}{2} \left(\frac{1}{2} \cdot \frac{3}{4} \right) + \frac{1}{2} = -\frac{3}{16} + \frac{8}{16} = \frac{5}{16} \end{aligned}$$

6.3 a) $f = xy + xz - yz$
 $= \underline{x^T A x}$

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$D_1 = 0$$

$$D_2 = -1/4 < 0$$

$$D_3 = \frac{1}{2} \left(-\frac{1}{4} \right) + \frac{1}{2} \left(-\frac{1}{4} \right) = -1/4$$

indefn.

no global max/min

$$f'(\underline{x}) = 2A\underline{x} = \underline{0}$$

$$-1 \cdot 1 + 1 \cdot (-1) = -2$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \xrightarrow{I} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{I} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{I} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Stat pts: $\underline{x} = \underline{0}$
saddle pt

$$H(f)(\underline{0}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \text{ indefn. } D_2 = -1 < 0$$

b) $f = x^2 + y^2 + z^2 + w^2$
 $+xy + yz + zw = \underline{x^T A x}$

A pos. defn
 (from 2d)

$$f_{\min} = \underline{0}$$

"

$$f(0, 0, 0, 0)$$

Stat pts: $2A\underline{x} = \underline{0}$

$$A\underline{x} = \underline{0}$$

$$\underline{x} = \underline{0}$$

local min

$$H(f)(\underline{0}) = 2A$$

pos. defn

$\Rightarrow$ local min

c) $f = x^4 + y^4 + z^4 + z^2$

$$f'_x = 4x^3 = 0$$

$$f'_y = 4y^3 = 0$$

$$f'_z = 4z^3 + 2z = 0$$

$$H(f) = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 + 2 \end{pmatrix}$$

$$H(f)(\underline{0}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

pos. semidefn.

$$z = 0, 1, 2 \geq 0$$

no concl.

Stat pts:

$$x = 0, y = 0$$

$$2z(2z^2 + 1) = 0$$

$$z = 0 \text{ or } 2z^2 + 1 = 0$$

$$\left. \begin{matrix} (x, y, z) = \\ (0, 0, 0) \end{matrix} \right\}$$

$$\text{Alt: } f(0, 0, 0) = \underline{0}$$

$$f(x, y, z) \geq 0 \text{ for all } (x, y, z)$$

$$f_{\min} = \underline{0} \quad (0, 0, 0) \text{ local/global min.}$$

6.4 a) $f(x) = x^T A x$ $H(x) = 2A$ pos. defn.
 A pos. defn.

A and $2A$ has the same definiteness

f convex

b) $f = e^{x-2y+z} = e^u$, $u = x-2y+z$

$$\begin{aligned} f'_x &= \frac{e^u \cdot 1}{e^u} \\ f'_y &= e^u \cdot (-2) \\ f'_z &= \frac{e^u \cdot 1}{e^u} \end{aligned} \quad H(f) = \begin{pmatrix} e^u \cdot 1 \cdot 1 & e^u \cdot 1 \cdot (-2) & e^u \cdot 1 \cdot 1 \\ e^u \cdot 1 \cdot (-2) & e^u \cdot (-2) \cdot (-2) & e^u \cdot (-2) \cdot 1 \\ e^u \cdot 1 \cdot 1 & e^u \cdot (-2) \cdot 1 & e^u \cdot 1 \cdot 1 \end{pmatrix}$$

$$= e^u \cdot \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\begin{aligned} R(3) &= R(1) \\ R(2) &= -2R(1) \end{aligned}$$

$H(f)$ pos. semidefn
for all x, y, z

f convex

$$\begin{aligned} D_1 &= 1 \\ D_2 &= 0 \\ D_3 &= 0 \end{aligned}$$

RRC: $rk=1$
pos. semidefn

c) $f = \frac{xy+xz+yz}{xyz}$, $x, y, z > 0$

$$= \frac{xy}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz}$$

$$= \frac{1}{z} + \frac{1}{y} + \frac{1}{x}$$

f convex
↑

$$f'_x = -x^{-2}$$

$$f'_y = -y^{-2}$$

$$f'_z = -z^{-2}$$

$$H(f) = \begin{pmatrix} 2/x^3 & 0 & 0 \\ 0 & 2/y^3 & 0 \\ 0 & 0 & 2/z^3 \end{pmatrix} \quad \begin{aligned} \lambda_1 &= 2/x^3 > 0 \\ \lambda_2 &= 2/y^3 > 0 \\ \lambda_3 &= 2/z^3 > 0 \end{aligned}$$

6.5 Find $V(f) = \text{range of } f$

a) $f = \ln(1 + \underbrace{2x^2 + 2xy + 3y^2 - 2xz + z^2})$

$$= \ln(1+u), \quad u = 2x^2 + 2xy + 3y^2 - 2xz + z^2$$

$$\underline{x}^T A \underline{x}$$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$V(u) = [0, \infty) \iff u_{\min} = \underline{\underline{0}} \text{ at } (4, 0, 0) \leftarrow$$



$$D_1 = 2$$

$$D_2 = 5$$

$$D_3 = -1 \cdot 3 + 1 \cdot 5 = 2$$

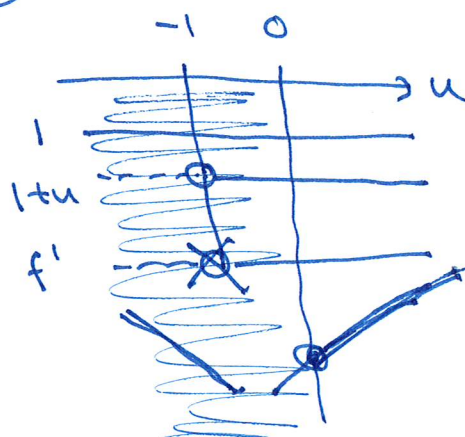
pos. defn.

Look at $f(u) = \ln(1+u)$, $u \geq 0$
 $f'(u) = \frac{1}{1+u} \cdot 1 = \frac{1}{1+u}$
 increasing when $u \geq 0$

$$f_{\min} = f(0) = \ln(1+0) = \underline{\underline{0}}$$

$$f(u) \rightarrow \infty \text{ when } u \rightarrow \infty$$

$$V(f) = \underline{\underline{[0, \infty)}}$$



$$b) f = (x^2 + y^2 + z^2) \cdot e^{-x^2 - y^2 - z^2} = u \cdot e^{-u}, \quad u = x^2 + y^2 + z^2$$

quadr. form
pos. defn.

$$f(u) = u e^{-u}$$

$$\begin{aligned} f'(u) &= 1 \cdot e^{-u} + u \cdot e^{-u} \cdot (-1) \\ &= e^{-u} - u e^{-u} \\ &= (1-u) e^{-u} \end{aligned}$$

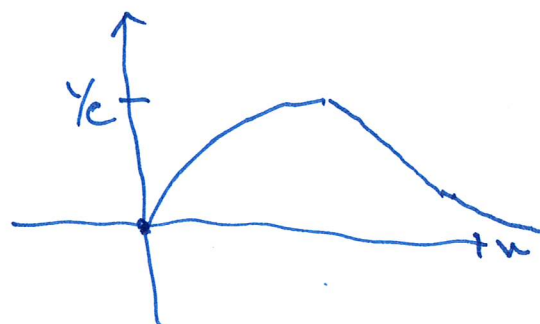
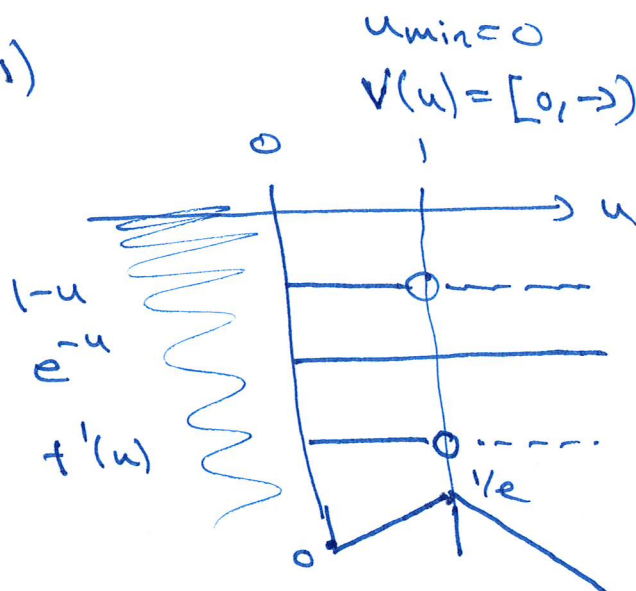
$$f_{\max} = f(1) = 1 \cdot e^{-1} = \underline{\underline{1/e}}$$

$$f(0) = 0$$

$$u \rightarrow \infty: f(u) = u e^{-u} = \frac{u}{e^u} \rightarrow 0$$

$$f_{\min} = f(0) = \underline{\underline{0}}$$

$$V(f) = \underline{\underline{[0, 1/e]}}$$



7.2b) max/min $f = x^2 + y^2 + z^2$ when $3x^2 + 2y^2 + 2z^2 = 12$

$$h = x^2 + y^2 + z^2 - \lambda(3x^2 + 2y^2 + 2z^2 - 12)$$

$$\begin{aligned} h'_x &= 2x - \lambda \cdot 6x = 0 \\ h'_y &= 2y - \lambda \cdot 4y = 0 \\ h'_z &= 2z - \lambda \cdot 4z = 0 \\ 3x^2 + 2y^2 + 2z^2 &= 12 \end{aligned}$$

$$2x(1-3\lambda) = 0$$

$$2y(1-2\lambda) = 0$$

$$2z(1-2\lambda) = 0$$

$$x=0 \text{ or } \lambda = 1/3$$

$$y=0 \text{ or } \lambda = 1/2$$

$$z=0 \text{ or } \lambda = 1/2$$

i) $\lambda = 1/3$

$$y = z = 0$$

$$3x^2 = 12$$

$$x^2 = 4$$

$$x = \pm 2$$

$$(\pm 2, 0, 0; 1/3) \quad f = 4$$

ii) $\lambda = 1/2$

$$x = 0$$

$$2y^2 + 2z^2 = 12$$

$$y^2 + z^2 = 6$$

$$(0, y, z; 1/2) \quad f = 6$$

with $y^2 + z^2 = 6$

iii) λ is not $1/2, 1/3$

$$x = y = z = 0$$

$$0 \neq 12$$

no points

Note: i) $3x^2 + 2y^2 + 2z^2 = 12$ is compact $\xRightarrow{\text{EVT}}$ there is a max/min
 $-2 \leq x \leq 2$
 $-\sqrt{6} \leq y, z \leq \sqrt{6}$

ii) Adm pts where NDCQ fails: None

NDCQ: $\text{rk} \begin{pmatrix} 6 & 4 & 4 \end{pmatrix} = 1$

NDCQ fails: $6x = 4y = 4z = 0$

$$(x, y, z) = (0, 0, 0)$$

$0 \neq 12$ not adm.

Concl:

$$f_{\max} = \underline{\underline{6}}$$

$$f_{\min} = \underline{\underline{4}}$$

Key Problems

Problem 1.

Check that the Markov chain with transition matrix A is regular, and find its equilibrium state $\mathbf{v}$:

$$\text{a) } A = \begin{pmatrix} 0.40 & 0.15 \\ 0.60 & 0.85 \end{pmatrix}$$

$$\text{b) } A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$$

$$\text{c) } A = \begin{pmatrix} 0.30 & 0.15 \\ 0.70 & 0.85 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 0.86 & 0.42 \\ 0.14 & 0.58 \end{pmatrix}$$

$$\text{e) } A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$$

$$\text{f) } A = \begin{pmatrix} 0.2 & 0.4 & 0 \\ 0.8 & 0.4 & 0.7 \\ 0 & 0.2 & 0.3 \end{pmatrix}$$

Problem 2.

Determine the definiteness of the symmetric matrix:

$$\text{a) } A = \begin{pmatrix} 7 & 4 \\ 4 & 3 \end{pmatrix}$$

$$\text{b) } A = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\text{c) } A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & 7 & 0 \\ -5 & 0 & 35 \end{pmatrix}$$

$$\text{e) } A = \begin{pmatrix} -1 & -2 & -2 \\ -2 & -4 & -4 \\ -2 & -4 & -2 \end{pmatrix}$$

Problem 3.

Find the symmetric matrix of the quadratic form, and determine its definiteness:

$$\text{a) } f(x,y) = x^2 - 8xy + 3y^2$$

$$\text{b) } f(x,y,z) = 2x^2 - 2xz + 3y^2 + z^2$$

$$\text{c) } f(x,y,z) = 3x^2 + 4xy - 4xz + 3y^2 + 4yz + 8z^2$$

$$\text{d) } f(x,y,z,w) = xw - yz$$

Problem 4.

Determine the definiteness of the symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Problem 5.

Find an orthogonal matrix P such that $P^T A P = D$ is a diagonal matrix:

$$\text{a) } A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\text{b) } A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\text{c) } A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Exercise Problems

Problems from the textbook: [E] 4.8 - 4.17

Exam problems: Midterm exam 10/2018 Question 1-7

Key Problems

Problem 1.

We consider the quadratic form $f(x,y,z,w) = x^2 + ay^2 - 2ayz + az^2 + w^2 - 2xw$ with parameter a . Determine the definiteness of f when

a) $a > 0$

b) $a = 0$

c) $a < 0$

Problem 2.

Solve the unconstrained optimization problem $\max / \min f(\mathbf{x})$:

a) $f(x,y,z) = 5x^2 + 6xy + 2y^2 + 16xz + 10yz + 13z^2$

b) $f(x,y,z,w) = 2xz - 2yw - x^2 - y^2 - z^2 - w^2$

c) $f(x,y,z,w) = 2xy + 2xz + 2yw + 2zw$

d) $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

Problem 3.

Find all stationary points of f , classify them as local maximum/minimum points or saddle points, and determine whether f has global maximum/minimum values:

a) $f(x,y,z) = xy + xz - yz$

b) $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

c) $f(x,y,z) = x^4 + y^4 + z^4 + z^2$

d) $f(x,y,z) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

Problem 4.

Determine whether f is a convex or concave function:

a) $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

b) $f(x,y,z) = e^{x-2y+z}$

c) $f(x,y,z) = x^4 + y^4 + z^4 + z^2$

d) $f(x,y,z) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

e) $f(x,y,z) = \frac{xy + xz + yz}{xyz}$ defined for $x,y,z > 0$

Problem 5.

Determine the range $V(f)$ of f :

a) $f(x,y,z) = \ln(1 + 2x^2 + 2xy + 3y^2 - 2xz + z^2)$

b) $f(x,y,z) = (x^2 + y^2 + z^2)e^{-x^2-y^2-z^2}$

Exercise Problems

Problems from the textbook: [E] 5.1 - 5.12

Exam problems: [Midterm 10/2017] Question 1-8

Key Problems

Problem 1.

Check if the given sets are compact (closed and bounded). It is useful to sketch the sets:

- a) $D = \{(x,y) : x,y \geq 0 \text{ and } 2x + 3y \leq 6\}$ b) $D = \{(x,y) : 4x^2 + 9y^2 \leq 36\}$
 c) $D = \{(x,y) : x,y \geq 1 \text{ and } 2x + 3y \geq 12\}$ d) $D = \{(x,y) : 4xy \leq 1 \text{ and } x,y > 0\}$

Problem 2.

Solve the Lagrange problems:

- a) $\max f(x,y,z) = x + 2y + 3z$ when $2x^2 + y^2 + 2z^2 = 9$
 b) $\max / \min f(x,y,z) = x^2 + y^2 + z^2$ when $3x^2 + 2y^2 + 2z^2 = 12$

Problem 3.

Use the second order condition to solve the Lagrange problem:

- a) $\max / \min f(x,y,z) = 4x^2 + 9y^2 + z^2$ when $x + y + z = 1$
 b) $\max / \min f(x,y,z,w) = xw - yz$ when $x^2 + 4y^2 = 4$ and $4z^2 + 9w^2 = 36$

Problem 4.

Determine if there are any admissible points such that the NDCQ fails when the constraints are given by:

- a) $xyz = 1$ b) $3x^2 + 3y^2 + 8z^2 = 1$
 c) $x^3 + y^3 + z^3 = 0$ d) $xy - zw = 1$ and $x + y + z + w = 4$

Exercise Problems

Problems from the textbook: [E] 6.1, 6.2, 6.3ab, 6.4, 6.11

Exam problems Final exam 11/2019 Question 4ab

Answers to Key Problems

Problem 1.

- a) Compact b) Compact
 c) Not compact (not bounded) d) Not compact (not bounded)

Problem 2.

- a) $f_{\max} = 9$ b) $f_{\min} = 4$ $f_{\max} = 6$

Problem 3.

- a) $f_{\min} = 36$ 49 b) $f_{\max} = 4$, $f_{\min} = -4$

Problem 4.

- a) None b) None c) $(x,y,z) = (0,0,0)$ d) None

Exam	Midterm exam in GRA 6035 Mathematics
Date	October 14th, 2022 at 0900 - 1000

This exam consists of 8 problems with score 0 - 3p each, and maximal score on this exam is 24p.
You must give reasons for your answers.

Question 1.

Determine the dimension of the null space of the matrix A :

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Question 2.

Write $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix},$$

Question 3.

Determine all values of t such that the matrix A has maximal rank:

$$A = \begin{pmatrix} 1 & 1 & t \\ t & 3 & 1 \\ 3 & 4 & 3 \end{pmatrix}$$

Question 4.

Determine the equilibrium state of the Markov chain with transition matrix A :

$$A = \begin{pmatrix} 0.72 & 0.07 \\ 0.28 & 0.93 \end{pmatrix}$$

Question 5.

Determine the definiteness of the quadratic form $q(x, y, z) = -x^2 + 4xy + 2xz - 4y^2 - 4yz - z^2$.

Question 6.

Determine the dimension of the eigenspace E_λ of A which contains the vector $\mathbf{v}$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 7.

Determine the scalar a such that $\mathbf{v} - a \cdot \mathbf{w}$ is orthogonal to $\mathbf{w}$ when $\mathbf{v} = (1, 0, 4, 3)$ and $\mathbf{w} = (1, 1, 1, 7)$.

Question 8.

The points $(1, 1, 3, 4)$ and $(0, 3, 1, 2)$ are solutions of a 3×4 linear system $A\mathbf{x} = \mathbf{b}$, where the minor $M_{123,124} = 2$. Determine all solutions of the linear system of the form $(x, y, z, 0)$.