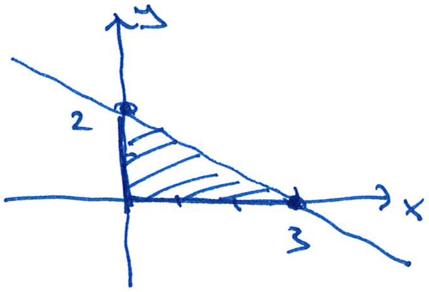
	Plan	
1	Key Problems:	7.1, 7.3b, 7.4, 8.1b, 8.2, 9.2, 9.3, 9.4
2	Final exams:	04/2022 Q3

7.1

a) D: $x, y \geq 0$ $2x+3y \leq 6$



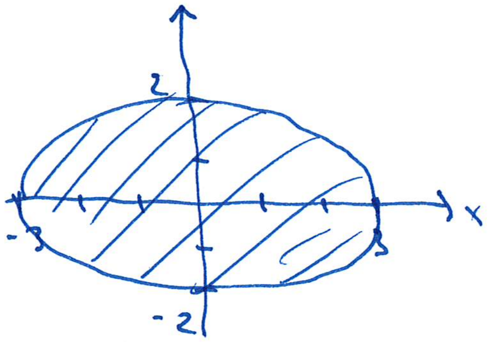
$2x+3y=6$

$2x+3y<6$ $\quad \frac{3y}{3} < \frac{6-2x}{3}$

$0 \leq x \leq 3$
$0 \leq y \leq 2$

D closed and bounded

b) D: $4x^2+9y^2 \leq 36$

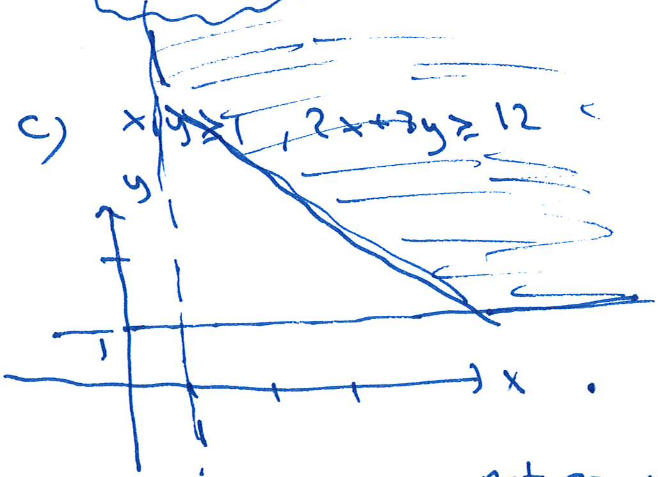


$4x^2+9y^2=36$ $\quad 1:36$

$\frac{x^2}{9}+\frac{y^2}{4}=1$ ellipse

compact $\quad -3 \leq x \leq 3$
$-2 \leq y \leq 2$

c) $x, y \geq 1$, $2x+3y \geq 12$

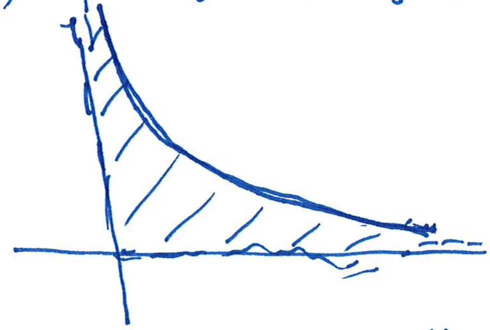


$2x+3y=12$

not compact

not bounded

d) D: $x, y > 0$, $4xy \leq 1$



$4xy=1 \Leftrightarrow y=\frac{1}{4x}=\frac{1/4}{x}$

not closed

not bounded

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7.3b) See lecture 7.

7.4 a) $xyz = 1$: NDCQ: $\text{rk}\begin{pmatrix} yz & xz & xy \end{pmatrix} = 1$

$g = xyz$ NDCQ fails: $\text{rk} = 0$, ie $yz = xz = xy = 0$

not adm. pts

d) $\left.\begin{matrix} xy - zw = 1 \\ x + y + z + w = 4 \end{matrix}\right\}$: NDCQ $\text{rk}\begin{pmatrix} y & x & -w & -z \\ 1 & 1 & 1 & 1 \end{pmatrix} = 2$

NDCQ fails: all 2-minors are zero

$y - x = 0$ $x + w = 0$ $-w + z = 0$

$y = x$ $w = -x$ $z = w = -x$

$\Rightarrow (x, x, -x, -x)$ have $\text{rk} < 2$

not adm.: $x + x + (-x) + (-x) = 4$

$0x = 4$

impossible

8.1 b) max $f = xz + yw$ when $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$

KT-prob in std. form

D compact

$$\mathcal{L} = xz + yw - \lambda_1(x^2+y^2-1) - \lambda_2(4z^2+9w^2-36)$$

FOC:
$\mathcal{L}'_x = z - \lambda_1 \cdot 2x = 0$ (1)
$\mathcal{L}'_y = w - \lambda_1 \cdot 2y = 0$ (2)
$\mathcal{L}'_z = x - \lambda_2 \cdot 8z = 0$ (3)
$\mathcal{L}'_w = y - \lambda_2 \cdot 18w = 0$ (4)

C: $x^2+y^2 \le 1$
$4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$
$\lambda_1 \cdot (x^2+y^2-1) = 0$
($\lambda_1 = 0$ if $x^2+y^2 < 1$)

$\lambda_2 \ge 0$
$\lambda_2(4z^2+9w^2-36) = 0$
($\lambda_2 = 0$ if $4z^2+9w^2 < 36$)

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and

(1) $z = 2\lambda_1 x$
(3) $x - 8\lambda_2(2\lambda_1 x) = 0$
$x \cdot (1 - 16\lambda_1\lambda_2) = 0$
$x = 0$ or $\lambda_1\lambda_2 = 1/16$

(2) $w = 2\lambda_1 y$
(4) $y - 18\lambda_2(2\lambda_1 y) = 0$
$y(1 - 36\lambda_1\lambda_2) = 0$
$y = 0$ or $\lambda_1\lambda_2 = 1/36$

Cases:
a) $x = 0$ and $y = 0$
b) $x = 0$ and $\lambda_1\lambda_2 = 1/36$
c) $y = 0$ and $\lambda_1\lambda_2 = 1/16$

(a) $x=0, y=0$:
$z=0, w=0$
$x^2+y^2 < 1$ $\lambda_1 = 0$
$4z^2+9w^2 < 36$ $\lambda_2 = 0$
$(0,0,0,0;0,0)$ $f=0$

(b) $x=0, \lambda_1\lambda_2 = 1/36$:
$z = 0$, $\boxed{w = 2\lambda_1 y}$ $\to \lambda_1 = \frac{w}{2y}$
$\lambda_1 > 0$: $x^2+y^2 = 1$ $y^2 = 1$ $y = \pm 1$
$\lambda_2 > 0$: $4z^2+9w^2 = 36$ $9w^2 = 36$ $w^2 = 4$ $w = \pm 2$
$(0,1,0,2;1,1/36)$ $f=2$ $(0,-1,0,-2;1,1/36)$ $f=2$

(c) $y=0,\ \lambda_1\lambda_2 = 1/16$:

$w=0 \qquad z = 2\lambda_1 x \Rightarrow \lambda_1 = \frac{z}{2x}$

$\lambda_1>0 \qquad x^2+y^2=1 \qquad x^2=1 \qquad \underline{x=\pm 1}$

$\lambda_2>0 \qquad 4z^2+9w^2=36 \qquad 4z^2=36 \quad z^2=9 \quad \underline{z=\pm 3}$

$(1,0,3,0;\ 3/2,\ 1/24)$ with $f=3$ $\qquad (-1,0,-3,0;\ 3/2,\ 1/24)$ with $f=3$

$\underline{f_{max}=3}$ at $(1,0,3,0)$, $(-1,0,-3,0)$ with $\lambda_1 = 3/2$, $\lambda_2 = 1/24$?

SOC:

$h = L(x,y,z,w;\ 3/2,\ 1/24)$

$= xz+yw-\frac{3}{2}(x^2+y^2-1)-\frac{1}{24}(4z^2+9w^2-36)$

$= 3+\underbrace{xz+yw-\frac{3}{2}x^2-\frac{3}{2}y^2-\frac{1}{6}z^2-\frac{3}{8}w^2}$

$$A=\begin{pmatrix}-3/2 & 0 & 1/2 & 0\\ 0 & -3/2 & 0 & 1/2\\ 1/2 & 0 & -1/6 & 0\\ 0 & 1/2 & 0 & -3/8\end{pmatrix}$$

$$H(h)=\begin{pmatrix}-3 & 0 & 1 & 0\\ 0 & -3 & 0 & 1\\ 1 & 0 & -1/3 & 0\\ 0 & 1 & 0 & -3/4\end{pmatrix}$$

$D_1=-3 \qquad \Delta_1 = -3,-3,-1/3,-3/4<0$

$D_2=9 \qquad \Delta_2 = 9,\ 1,\ 1/4,\ 0,\ 5/4,\ 9/4 \ge 0$

$D_3=-3\cdot 0=0 \qquad \Delta_3 = 0,\ -5/12,\ 0,\ -15/4 \le 0$

$D_4 = 1\cdot\begin{vmatrix}-3 & 1 & 0\\ 0 & 0 & 1\\ 1 & -1/3 & 0\end{vmatrix} - 3/4\cdot 0 = 0 \qquad \Delta_4 = 0 \ge 0$

$$\begin{vmatrix}-3 & 0 & 0\\ 0 & -3 & 1\\ 0 & 1 & -3/4\end{vmatrix} = -15/4$$

$$\begin{vmatrix}-3 & 1 & 0\\ 1 & -1/3 & 0\\ 0 & 0 & -3/4\end{vmatrix} = 0$$

h concave $\overset{SOC}{\Rightarrow}$ $\underline{f_{max}=3}$

EVT: $D:\ \begin{matrix}x^2+y^2\le 1\\ 4z^2+9w^2\le 36\end{matrix}$ is compact

$\overset{EVT}{\Rightarrow}$ there is a max

Either $\underline{f_{max}=3}$ or the max does not satisfy NDCQ.

NDCQ: $J=\begin{pmatrix}2x & 2y & 0 & 0\\ 0 & 0 & 8z & 18w\end{pmatrix}$

BB: rk $J=2$ $\quad$ rk $J<2$ if $x=y=0$ or $z=w=0$ No.

NN: no cond.

BN: rk$(2x\ 2y\ 0\ 0)=1$ fails if $x=y=0$

NB: rk$(0\ 0\ 8z\ 18w)=1$ " $z=w=0$

Concl:

NDCQ satisfied at all adm pts.

$\Rightarrow f_{max}=3$

8.2 a) max $f = x^2y^2z^2$ when $x^2+y^2+z^2+x^2y^2z^2 \le 4$

$L = x^2y^2z^2 - \lambda(x^2+y^2+z^2+x^2y^2z^2-4)$

Note: – std. form KT prob.
– D is compact
$x^2 \le 4$ $-2 \le x \le 2$
" y "
" z "

FOC: $L'_x = 2xy^2z^2 - \lambda(2x+2xy^2z^2) = 0$ (1)
$L'_y = 2yx^2z^2 - \lambda(2y+2yx^2z^2) = 0$ (2)
$L'_z = 2zx^2y^2 - \lambda(2z+2zx^2y^2) = 0$ (3)

C: $x^2+y^2+z^2+x^2y^2z^2 < 4$ or $x^2+y^2+z^2+x^2y^2z^2 = 4$

CSC: $\lambda = 0$ | $\lambda \ge 0$

$2xy^2z^2 = 0$
$2yx^2z^2 = 0$
$2zx^2y^2 = 0$
$x=0$ or $y=0$ or $z=0$

$\Rightarrow$ many cand. pts with $f=0$

(1) $2x\cdot(y^2z^2 - \lambda - \lambda y^2z^2) = 0$

$x=0$ or $y^2z^2 = \lambda + \lambda y^2z^2 = \lambda\cdot(1+y^2z^2)$

$f=0$ | $\lambda = \dfrac{y^2z^2}{1+y^2z^2}$

(2) $y=0$ or $x^2z^2 - \lambda - \lambda x^2z^2 = 0$

$f=0$ | $\lambda = \dfrac{x^2z^2}{1+x^2z^2}$

$$\frac{y^2z^2}{1+y^2z^2} = \frac{x^2z^2}{1+x^2z^2}$$

$$y^2z^2 + \cancel{y^2z^2\cdot x^2z^2} = x^2z^2 + \cancel{x^2z^2\cdot y^2z^2}$$

$z^2(y^2-x^2) = 0$

$z=0$ or $y^2=x^2$

$f=0$ | (1)+(2): $y^2 = x^2$
(2)+(3): $z^2 = y^2$

Cand. pts. with $x,y,z \ne 0$:

$a = x^2=y^2=z^2$: $3a + a^3 = 4$
$a^3+3a-4 = 0$
$(a-1)(a^2+a+4) = 0$

$x^2 = 1$ or $\cancel{x^2 = \dfrac{-1 \pm \sqrt{1^2-16}}{2}}$

⇓

$x^2=1,\ y^2=1,\ z^2=1$. $\lambda = 1/2$

⇓

$(\pm1, \pm1, \pm1;\ 1/2)$
$f=1$

$\overset{\text{EVT}}{\Longrightarrow}$ $f_{max} = 1$
NDCQ

NDCQ: $\underbrace{x^2+y^2+z^2+x^2y^2z^2}_{g}=4$ $\qquad rk\,(g'_x \;\; g'_y \;\; g'_z)=1$

$g'_x=g'_y=g'_z=0$:

$x=0 \Leftrightarrow 2x\cdot(1+y^2z^2)=0$

$y=0 \qquad 2y\,(1+x^2z^2)=0$

$z=0 \qquad 2z\,(1+x^2y^2)=0$

Concl:
NDCQ satisfied at all adm pts.

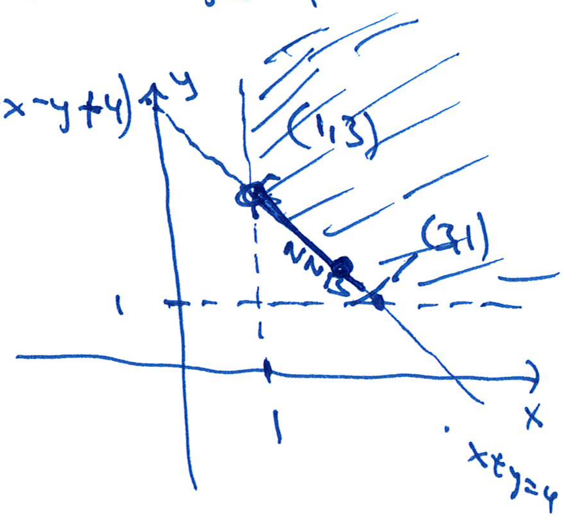
b) max $f=\ln(x^2y)-x-y$ when $\begin{cases} x\geq 1 & \Leftrightarrow \; -x\leq -1 \\ y\geq 1 & \quad\;\; -y\leq -1 \\ x+y\geq 4 & \quad\;\; -x-y\leq -4 \end{cases}$

$=2\ln x+\ln y-x-y$

$L=2\ln x+\ln y-x-y-\lambda_1(-x+1)-\lambda_2(-y+1)-\lambda_3(-x-y+4)$

FOC: $L'_x=\dfrac{2}{x}-1+\lambda_1+\lambda_3=0$

$L'_y=\dfrac{1}{y}-1+\lambda_2+\lambda_3=0$



C+CSC: Many cases

BBB: not possible

BBN: —"—

BNB: (1,3) $\qquad$ NBB: (3,1)

NNB: $x+y=4$, $1\leq x\leq 3$

⋮

D is not compact

Look at: $x+y=4$, $x>1, y>1$ $\quad \lambda_1=\lambda_2=0$

$\left.\begin{array}{l} \dfrac{2}{x}-1+\lambda_3=0 \\ \dfrac{1}{y}-1+\lambda_3=0 \end{array}\right\}$ $\lambda_3=1-\dfrac{2}{x}=1-\dfrac{1}{y}$

$\dfrac{2}{x}=\dfrac{1}{y} \qquad x=2y$

$\left.\begin{array}{l} x+y=4 \\ 2y+y=4 \\ 3y=4 \\ y=4/3 \\ x=8/3 \end{array}\right\}$

$\lambda_3=1-3/4=1/4$

Cand pt:
$(8/3, 4/3;\ 0, 0, 1/4)$

SOC:
$h=2\ln x+\ln y-x-y-\frac{1}{4}(-x-y+4)$

$H(h)=\begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$

h concave $\Rightarrow$ SOC

$f_{max}=f(8/3, 4/3)$

$f_{max}=\underline{8\ln 2-3\ln 3-4}$

9.2 a) $f = \underline{x^2 + 4xz + y^2 + 5z^2} \; \underline{-4y + 2z}$

$= \underline{x}^T A \underline{x} + B\underline{x}$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{pmatrix} \qquad B = (0 \;\; -4 \;\; 2)$$

$$\left.\begin{array}{l} D_1 = 1 \\ D_2 = 1 \\ D_3 = 1 \cdot (5-4) = 1 \end{array}\right\} \begin{array}{l} A \text{ pos. defin.} \\ \Rightarrow \text{any stat. pt is } \underline{\text{min}} \end{array}$$

$$f'(\underline{x}) = 2A\underline{x} + B^T = \underline{0}$$

$$2A\underline{x} = -B^T$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{pmatrix} \underline{x} = \quad A\underline{x} = -\tfrac{1}{2}B^T = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 5 & -1 \end{array}\right) \xrightarrow{-2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \qquad \begin{array}{l} x = 2 \\ \cancel{x} = 2 \\ z = -1 \end{array}$$

$$f_{min} = f(2,2,-1) = \cancel{4 - 8 + 4} + 5 - 8 - 2 = -5$$

$$V_f = \underline{\underline{[-5, \rightarrow)}}$$

9.3. min $f = \underline{x}^T A \underline{x}$ when $x^2+y^2+z^2+w^2 = 6$

$\underline{x}^T \cdot I \cdot \underline{x} = 6$ $\quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

a) $A = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix}$

$D_1 = -4$

$D_2 = 40$

$D_3 = -4 \cdot 46 + 2 \cdot 20$
$= -184 + 40 = -144$

$D_4 = 0$

$= \begin{vmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{vmatrix} \begin{matrix} +1/2 \\ +1/2 \end{matrix} = \begin{vmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 0 & -2 & -4 & 4 \\ 0 & 2 & 4 & -4 \end{vmatrix}$

$= -4 \cdot [-10 \cdot 0 + 2 \cdot 0 + 2 \cdot 0] = 0$

A negative semidefin.

$\Rightarrow$ f concave

b) $L = \underline{x}^T A \underline{x} - \lambda(\underline{x}^T I \underline{x} - 6)$

FOC: $L'(\underline{x}) = 2A\underline{x} - \lambda \cdot (2I\underline{x}) = \underline{0}$

$2A\underline{x} - 2\lambda I \underline{x} = \underline{0} \quad |:2$

$A\underline{x} - \lambda I \underline{x} = 0$

$(A - \lambda I)\underline{x} = \underline{0}$

$\cancel{\underline{x} = \underline{0}}$ or $\underline{x}$ is in E_λ (λ eigenvalue of A)

C: $\underline{x}^T I \underline{x} = 6 \quad \underline{x}^T \underline{x} = 6 \quad \|\underline{x}\|^2 = 6$

$\lambda = -12$: $(A + 12I)\underline{x} = \underline{0}$ $\quad \begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 2 & -2 & 2 \\ 2 & -2 & 7 & 3 \\ 2 & 2 & 3 & 7 \end{pmatrix} \underline{x} = \underline{0}$

$(x, y, z, w; \lambda) = (0, -2w, -w, w; -12)$

Solve with Gauss.

C: $0 + 4w^2 + w^2 + w^2 = 6$
$6w^2 = 6$
$w^2 = 1 \quad w = \pm 1$

$\Rightarrow (0, -2, -1, 1; -12) \quad (0, 2, 1, -1; -12)$

c) Candidates for max:

$\cancel{\underline{x}=\underline{0}}$ or $(A-\lambda I)\underline{x}=\underline{0}$ and $\underline{x}^T\underline{x}=6$

Assume that λ is an eigenvalue of A, and $\underline{x}$ is an eigenvector in E_λ such that $\underline{x}^T\underline{x}=6$.

$$f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{x}^T \lambda \underline{x} = \lambda\, \underline{x}^T\underline{x} = \lambda\cdot 6 = 6\lambda$$

min $\Leftrightarrow$ smallest eigenvalue

max $\Leftrightarrow$ biggest eigenvalue

Note: A neg. semidefin. with $|A|=0$ $\quad \underset{\underset{-12}{\shortparallel}}{\lambda_1}, \lambda_2, \lambda_3, \underset{\underset{0}{\shortparallel}}{\lambda_4} \leq 0$

$\Rightarrow f_{max} = \underline{\underline{0}}$ since $\lambda=0$ largest eigenvalue

See Final Exam 01/2020, Q4 for full solutions.

9.4. See Final Exam 12/2021 Q3 for full solutions

Key Problems

Problem 1.
Check if the given sets are compact (closed and bounded). It is useful to sketch the sets:

a) $D = \{(x,y) : x,y \geq 0 \text{ and } 2x + 3y \leq 6\}$
b) $D = \{(x,y) : 4x^2 + 9y^2 \leq 36\}$
c) $D = \{(x,y) : x,y \geq 1 \text{ and } 2x + 3y \geq 12\}$
d) $D = \{(x,y) : 4xy \leq 1 \text{ and } x,y > 0\}$

Problem 2.
Solve the Lagrange problems:

a) $\max f(x,y,z) = x + 2y + 3z$ when $2x^2 + y^2 + 2z^2 = 9$
b) $\max / \min f(x,y,z) = x^2 + y^2 + z^2$ when $3x^2 + 2y^2 + 2z^2 = 12$

Problem 3.
Use the second order condition to solve the Lagrange problem:

a) $\max / \min f(x,y,z) = 4x^2 + 9y^2 + z^2$ when $x + y + z = 1$
b) $\max / \min f(x,y,z,w) = xw - yz$ when $x^2 + 4y^2 = 4$ and $4z^2 + 9w^2 = 36$

Problem 4.
Determine if there are any admissible points such that the NDCQ fails when the constraints are given by:

a) $xyz = 1$
b) $3x^2 + 3y^2 + 8z^2 = 1$
c) $x^3 + y^3 + z^3 = 0$
d) $xy - zw = 1$ and $x + y + z + w = 4$

Exercise Problems

Problems from the textbook:	[E] 6.1, 6.2, 6.3ab, 6.4, 6.11
Exam problems	Final exam 11/2019 Question 4ab

Answers to Key Problems

Problem 1.
a) Compact
b) Compact
c) Not compact (not bounded)
d) Not compact (not bounded)

Problem 2.
a) $f_{\max} = 9$
b) $f_{\min} = 4$ $f_{\max} = 6$

Problem 3.
a) $f_{\min} = 36/49$
b) $f_{\max} = 4$, $f_{\min} = -4$

Problem 4.
a) None
b) None
c) $(x,y,z) = (0,0,0)$
d) None

Key Problems

Problem 1.
Solve the Kuhn-Tucker problems:

a) $\max f(x,y,z) = x - 2y + z$ when $x^2 + y^2 + z^2 \leq 3$
b) $\max f(x,y,z,w) = xz + yw$ when $x^2 + y^2 \leq 1$ and $4z^2 + 9w^2 \leq 36$

Problem 2.
Solve the Kuhn-Tucker problems:

a) $\max f(x,y,z) = x^2y^2z^2$ when $x^2 + y^2 + z^2 + x^2y^2z^2 \leq 4$
b) $\max f(x,y) = \ln(x^2y) - x - y$ when $x \geq 1,\ y \geq 1$ and $x + y \geq 4$

Exercise Problems

Problems from the textbook:	[E] 5.13 - 5.14, 6.3cd, 6.5 - 6.10
Exam problems:	[Final 11/2017] Question 3-4
	[Final 11/2018] Question 3-4

Answers to Key Problems

Problem 1.
a) $f_{\max} = 3\sqrt{2}$
b) $f_{\max} = 3$

Problem 2.
a) $f_{\max} = 1$
b) $f_{\max} = 8\ln 2 - 3\ln 3 - 4$

Key Problems

Problem 1.
We consider the constrained optimization problem $\max f(x,y,z) = 2x^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \leq 16$.

a) Find the maximum point and maximum value of f.

b) Use the envelope theorem to estimate the new maximum value of f when we change
i) the constraint to $x^4 + y^4 + z^4 \leq 20$ ii) the objective function to $f(x,y,z) = x^2 - 4y^2 - 2z^2$

Problem 2.
Determine the range of the following quadratic functions:

a) $f(x,y,z) = x^2 + 4xz + y^2 + 5z^2 - 4y + 2z$

b) $f(x,y,z,w) = 3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 6z^2 + 3w^2 + 1$

Problem 3.
We consider the Lagrange problem given by

$$\min\ f(x,y,z,w) = -4x^2 - 10y^2 - 5z^2 - 5w^2 + 4xz + 4xw - 4yz + 4yw + 6zw \text{ when } x^2 + y^2 + z^2 + w^2 = 6$$

a) Determine whether f is convex or concave.

b) Find all points (x,y,z,w) such that $(x,y,z,w;\lambda)$ satisfy the Lagrange conditions when $\lambda = -12$.

c) Solve $\max f(x,y,z,w)$ subject to $x^2 + y^2 + z^2 + w^2 = 6$.

Problem 4.
Let $g(x,y,z,w) = 3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 7z^2 + 4w^2$, and consider the Kuhn-Tucker problem given by

$$\max\ f(x,y,z) = x + y + z + w \text{ subject to } g(x,y,z,w) \leq 18$$

a) Determine the definiteness of the quadratic form g.

b) Write down the Kuhn-Tucker conditions of the problem in matrix form.

c) Write down the non-degenerate constraint qualification in this problem, and find all admissible points where this condition does not hold (if there are any).

d) Solve the Kuhn-Tucker problem.

e) Determine whether the set $D = \{(x,y,z,w) : g(x,y,z,w) \leq 18\}$ of admissible points is a compact set.

Exercise Problems

Exam problems [Final 01/2018] Question 1,3,4

Exam	Final exam in GRA 6035 Mathematics
Date	April 29th, 2022 at 1300 - 1600

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.
We consider the matrix A given by

$$A = \begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix}$$

(a) **(6p)** Compute the determinant and rank of A.
(b) **(6p)** Find a base of $\text{Null}(A - I)$.
(c) **(6p)** Show that $\lambda = 1, \lambda = -5$ and $\lambda = -1$ are eigenvalues of A.
(d) **(6p)** Find a matrix P such that $P^{-1}AP$ is diagonal, if it is possible.

Question 2.
(a) **(6p)** Solve the differential equation $y'' - 3y' + 2y = 6e^{-t}$.
(b) **(6p)** Solve the differential equation $ty' + y = 1$.
(c) **(6p)** Solve the differential equation $2ty' + y^2 = 1$.
(d) **(6p)** Find the equilibrium state of the system of differential equations. Is it stable?

$$\mathbf{y}' = \begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} \cdot \mathbf{y} + \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

Question 3.
We consider the function f given by $f(x, y, z) = 4y - 2x^2 - 3z^2 - 2xy - 8xz$ and the Lagrange problem given by

$$\max\ f(x, y, z) \text{ subject to } g(x, y, z) = x^2 + y^2 + 4z^2 + 4yz = 2$$

(a) **(6p)** Find all stationary points of f and classify them.
(b) **(6p)** Write down the Lagrange conditions of the Lagrange problem.
(c) **(6p)** Find all points $(x, y, z; \lambda)$ with $\lambda = 1$ that satisfy the Lagrange conditions.
(d) **(6p)** Solve the Lagrange problem.
(e) **Extra credit (6p)** Find a linear change of variables $\mathbf{x} = P\mathbf{u}$ such that the constraint $g(\mathbf{x}) = 2$ takes the form $\lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2 = 2$, and use this to describe the set of admissible points.