	Plan	
1	Key Problems:	10.1c, 10.2bc, 10.3b, 10.4bc, 11.2bcd, 11.3b, 12.1c, 12.2, 12.3, 12.4
2	Final exams:	12/2021 Q2

① Key Problems

10.1c $y' = t\sqrt{t^2+1}$

$$y = \int t\sqrt{t^2+1}\,dt = \int \cancel{t}\cdot\sqrt{u}\cdot\frac{1}{2\cancel{t}}\,du$$

$$\boxed{\begin{matrix} u = t^2+1 \\ du = 2t\,dt \end{matrix}}$$

$$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{1}{3}(t^2+1)^{3/2} + C$$

10.2b $y' = y^2 \cdot t$ sep. y^{-2}

$$\frac{1}{y^2}y' = t \qquad \int \frac{1}{y^2}\,dy = \int t\,dt$$

$$-\frac{1}{y} = \frac{1}{-1}y^{-1} = \frac{1}{2}t^2 + C$$

$$\frac{1}{y} = -\frac{1}{2}t^2 - C = \frac{-t^2-2C}{2}$$

$$y = \frac{2}{-t^2-2C}$$

10.2c) $y' = 5y\underbrace{(1-y/10)} = 5 \cdot y \cdot \frac{10-y}{10} = \underbrace{5}_{f(t)} \cdot \underbrace{\frac{y(10-y)}{10}}_{g(y)}$

$$\frac{10}{y(10-y)} y' = 5$$

$$\int \frac{10}{y(10-y)}\,dy = \int 5\,dt = 5t + C$$

Sep.

partial fractions:

$$\frac{10}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y}$$

$$10 = A(10-y) + By$$

$$10 = \underset{\overset{\shortparallel}{10}}{(10A)} + \underset{\overset{\shortparallel}{0}}{(-A+B)}y$$

$\underline{A=1}$ $\underline{B=1}$

$$\int \frac{1}{y} + \frac{1}{10-y}\,dy = 5t + C$$

$$\ln|y| - \ln|10-y| = 5t + C$$

$$\ln\frac{|y|}{|10-y|} = 5t + C \quad | \, e^{\cdot}$$

$$\frac{|y|}{|10-y|} = e^{5t+C} = e^{5t} \cdot e^{C}$$

$$\frac{y}{10-y} = \pm e^{C} \cdot e^{5t} = Ke^{5t} \quad | \cdot (10-y)$$

$$y = (10-y) \cdot Ke^{5t} = 10 \cdot Ke^{5t} - y \cdot Ke^{5t}$$

$$y + y \cdot Ke^{5t} = 10 \cdot Ke^{5t}$$

$$y(1 + Ke^{5t}) = 10 \cdot Ke^{5t} \qquad \underline{\underline{y = 10 \cdot \frac{Ke^{5t}}{1+Ke^{5t}}}}$$

10.3 b) $\underbrace{y' - 2ty}_{a(t)} = \underbrace{4t}_{b(t)}$ lin.

$$\int a(t)\,dt = \int -2t\,dt = -t^2 + C$$

$u = e^{-t^2}$ int. factor

$$y' - 2ty = 4t \quad | \cdot e^{-t^2}$$

$$(y \cdot e^{-t^2})' = 4te^{-t^2}$$

$$y \cdot e^{-t^2} = \int 4te^{-t^2}\,dt$$

$$= \int 4\cancel{t}e^u \left(\frac{1}{-2\cancel{t}}\right) du$$

$$\boxed{\begin{aligned} u &= -t^2 \\ du &= -2t\,dt \end{aligned}}$$

$$= -2\int e^u\,du$$

$$= -2e^u + C$$

$$y \cdot e^{-t^2} = -2e^{-t^2} + C \quad | \cdot e^{t^2}$$

$$y = \underline{\underline{-2 + Ce^{t^2}}}$$

$$y' = 4t + 2ty$$

$$= \underbrace{2t}_{f(t)} \cdot \underbrace{(2+y)}_{g(y)} \qquad \text{sep.}$$

$$\frac{1}{2+y}\,y' = 2t$$

$$\int \frac{1}{2+y}\,dy = \int 2t\,dt$$

$$\ln|2+y| = t^2 + C \quad | e^{\cdot}$$

$$|2+y| = e^{t^2+C} = e^{t^2} \cdot e^C$$

$$2+y = (\pm e^C)e^{t^2} = Ke^{t^2}$$

$$y = \underline{\underline{Ke^{t^2} - 2}}$$

10.4 b) $\underbrace{2y-3t^2}_{p}+\underbrace{2(y+t)}_{q}\,y'=0$

check if lin. / sep.

$(2y+2t)\,y' = 3t^2-2y$

$y' = \dfrac{3t^2-2y}{2y+2t}$ NO.

Exact? $h'_t = 2y-3t^2$

$h'_y = 2y+2t$

$h = 2yt - t^3 + C(y)$

$h'_y = (2yt - t^3 + C(y))'_y$

$= 2t - \cancel{0} + C'(y) = 2y+2t$

$C'(y) = 2y \qquad C(y) = y^2$

Yes, and $h = 2yt - t^3 + y^2$, so the general solution is:

$h = 2yt - t^3 + y^2 = C$

$y^2 + 2t\cdot y + (-t^3 - C) = 0$

$\Rightarrow y = \dfrac{-2t \pm \sqrt{4t^2 - 4\cdot 1(-t^3-c)}}{2}$

$= -t \pm \dfrac{\sqrt{4t^2+4t^3+4c}\cdot\frac{1}{2}}{2\cdot\frac{1}{2}} \qquad \frac{1}{2} = \frac{1}{\sqrt{4}}$

$= -t \pm \sqrt{t^2+t^3+c}$

In general: $\boxed{p + q\cdot y' = 0}$ is exact $\Rightarrow$ look for h s.t. $\begin{cases} h'_t = p \\ h'_y = q \end{cases}$

$\Uparrow$

$\dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial t}$

since

$p'_y = h''_{ty}$

$q'_t = h''_{yt}$

In this case:

$p = 2y - 3t^2 \quad q = 2y + 2t$

$p'_y = 2$

$q'_t = 2$

10.4 c) $\underbrace{y \cdot \frac{(1-2\ln t)}{t^3}}_{p} + \underbrace{\frac{\ln t}{t^2} \cdot y'}_{q} = 0$

Try to solve as an exact diff-eqn.

$h'_t = y \cdot \frac{(1-2\ln t)}{t^3}$

$h'_y = \frac{\ln t}{t^2} \longrightarrow h = \frac{\ln t}{t^2} \cdot y + C(t)$

$h'_t = \left(\frac{\ln t}{t^2}\right)'_t \cdot y + C'(t)$

$= \frac{\frac{1}{t} \cdot t^2 - \ln t \cdot 2t}{t^4} \cdot y + C'(t)$

$= \frac{t - 2t \cdot \ln t}{t^4} \cdot y + C'(t)$

$= \frac{1 - 2\ln t}{t^3} \cdot y + C'(t) = \frac{1-2\ln t}{t^3} \cdot y$

$C'(t) = 0 \qquad C(t) = 0$

Exact:

$h = \frac{\ln t}{t^2} \cdot y = C$

$y = C \frac{t^2}{\ln t}$

$y = \underline{\underline{\frac{Ct^2}{\ln t}}}$

Note:

$y \cdot \frac{(1-2\ln t)}{t^3} + \frac{\ln t}{t^2} \cdot y' = 0 \quad | \cdot t^3$

$y \cdot (1-2\ln t) + t \ln t \, y' = 0 \implies y' = \frac{y(2\ln t - 1)}{t \ln t}$

sep. and lin. but not so easy to solve in this way.

11.2 b) $y'' + 6y' + 9y = 4e^{-t}$

$$y = y_h + y_p = \underline{\underline{(C_1 + C_2 t)e^{-3t} + e^{-t}}}$$

y_h: $r^2 + 6r + 9 = 0$ $\qquad y_h = \underline{C_1 e^{-3t} + C_2 t e^{-3t}}$

$(r+3)^2 = 0$

$r_1 = r_2 = -3$

y_p: $y'' + 6y' + 9y = 4e^{-t}$

$$\left| \begin{array}{l} y = Ae^{-t} \\ y' = -Ae^{-t} \\ y'' = +Ae^{-t} \end{array} \right.$$

$Ae^{-t} + 6(-Ae^{-t}) + 9(Ae^{-t}) = 4e^{-t}$

$(A - 6A + 9A)e^{-t} = 4 \cdot e^{-t}$

$\underline{4A}\, e^{-t} = \underline{4}\, e^{-t}$

$4A = 4 \Rightarrow \underline{A = 1} \qquad y_p = \underline{1 \cdot e^{-t}}$

11.3 b) $y'' - 3y' + 2y = 3e^{2t}$

$$y = y_h + y_p = \underline{\underline{C_1 e^t + C_2 e^{2t} + 3te^{2t}}}$$

y_h: $r^2 - 3r + 2 = 0$ $\qquad y_h = \underline{C_1 e^t + C_2 e^{2t}}$

$(r-1)(r-2) = 0$

$\underline{r=1}, \underline{r=2}$

when $y = Ae^{2t}$ does not work multiply it with t

y_p: $y'' - 3y' + 2y = 3e^{2t}$ $\qquad y = \underline{Ate^{2t}}$

$$(4A+4At)e^{2t} - 3(A+2At)e^{2t} + 2(At)e^{2t} = 3e^{2t}$$

$$y' = A\cdot e^{2t} + At\cdot e^{2t}\cdot 2 = \underline{(A+2At)\cdot e^{2t}}$$

$$y'' = 2A\cdot e^{2t} + (A+2At)\cdot e^{2t}\cdot 2 = (2A+2A+4At)e^{2t} = \underline{(4A+4At)e^{2t}}$$

$$e^{2t}\cdot(4A + \cancel{4At} - 3A - \cancel{6At} + \cancel{2At}) = 3e^{2t}$$

$$A\cdot e^{2t} = 3e^{2t}$$

$$\underline{A=3}$$

$$y_p = 3te^{2t}$$

11.2 d) $y'' - y = t^2$ $\quad y = y_h + y_p = \underline{\underline{c_1e^t + c_2e^{-t} + t^2 - 2}}$

y_h: $y'' - y = 0$

$r^2 - 1 = 0$

$r = \pm 1$ $\quad y_h = \underline{c_1e^t + c_2e^{-t}}$

$\underline{y_p}$: $y'' - y = t^2$

$f = t^2$

$f' = 2t$

$f'' = 2$

$y = \underline{At^2 + Bt + C}$

$y' = 2At + B$

$y'' = 2A$

$$(2A) - (At^2 + Bt + C) = t^2$$

$$\underset{\substack{\shortparallel\\1}}{(-A)}t^2 + \underset{\substack{\shortparallel\\0}}{(-B)}t + \underset{\substack{\shortparallel\\0}}{(2A-C)} = t^2$$

$A=-1$ $\quad B=0$ $\quad C = 2A = -2$

$y_p = \underline{-t^2-2}$

11.3 b) $y = \dfrac{2}{1-t^2} = 2\cdot(1-t^2)^{-1}$

$y' = 2\cdot(-1)\underline{(1-t^2)^{-2}}\cdot(-2t) = \dfrac{4t}{(1-t^2)^2}$

$y' = \dfrac{4t}{(1-t^2)^2} = t\cdot\dfrac{4}{(1-t^2)^2} = \underline{t y^2}$ sep. non-lin.

12.1 c) $y_1' = y_1 + 4y_2 + 3$
$y_2' = y_1 - 2y_2 - 3$

$\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}\mathbf{y} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

lin. sys. of diff. eqn. nonhomogeneous

$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$

$\mathbf{y}' - A\mathbf{y} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$\mathbf{y}_h$: $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}\mathbf{y}$

$\mathbf{y} = \underline{C_1\cdot \mathbf{v}_1 e^{\lambda_1 t} + C_2\cdot \mathbf{v}_2 e^{\lambda_2 t}}$ if $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$ is diag. with

$= \underline{C_1\cdot\begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + C_2\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}}$ $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $P = (\mathbf{v}_1 | \mathbf{v}_2)$

Eigenvalues:

$\lambda^2 - \mathrm{tr}(A)\cdot\lambda + \det(A) = 0$

$\lambda^2 + \lambda - 6 = 0$

$\lambda = \dfrac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-6)}}{2}$

$= \dfrac{-1 \pm 5}{2} = 2, -3$

Eigenvectors:

$\lambda = 2$: $\begin{pmatrix} -1 & 4 \\ \cancel{1} & \cancel{-4} \end{pmatrix}$ $-x + 4y = 0$
$x = 4y$, y free

$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4y \\ y \end{pmatrix}$

$\lambda = -3$: $\begin{pmatrix} \cancel{4} & \cancel{4} \\ 1 & 1 \end{pmatrix}$ $x + y = 0$, y free

$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\underline{y} = \underline{y}_h + \underline{y}_p = \underline{\underline{c_1 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}}}$$

$\underline{y_p}$: $\quad \underline{y}' = A\underline{y} + \underline{b}$

$\underline{\text{Const sol'n}}$: $\underline{y}' = \underline{0}$

$A\underline{y} + \underline{b} = \underline{0}$

$A\underline{y} = -\underline{b}$

$$\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \cdot \underline{y} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 4 & -3 \\ 1 & -2 & 3 \end{array}\right) \rfloor \cdot 1 \rightarrow \left(\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -6 & 6 \end{array}\right)$$

$y_1 + 4y_2 = -3$

$-6y_2 = 6$

$y_1 = 4 - 3 = 1$

$y_2 = \underline{-1}$

$$\underline{y}_p = \underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$\underline{12.2.}$

a) $$\underbrace{\underline{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \underline{y}}, \quad \underline{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{y} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + c_3 \underline{v_3} e^{\lambda_3 t}$$

$$\begin{vmatrix} -5-\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = 0$$

$(-3-\lambda) \cdot \left((-5-\lambda)^2 - 1^2\right) = 0$

$\lambda = \underline{-3}$ or $\lambda^2 + 10\lambda + 24 = 0$

$\lambda = \underline{-4}, \lambda = \underline{-6}$

$\underline{\lambda = -3}$: $$\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$x - 2z = 0$

$-3z = 0$

y free

$y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = -4$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ \cancel{1} & \cancel{0} & \cancel{-1} \end{pmatrix}$ $z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = -6$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix}$ $z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$y = c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-6t}$$

$y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: $c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right)$$

$\underline{c_1 = 0}$

$c_2 - (-1/2) = 1$ $\quad c_2 = \underline{1/2}$

$2c_3 = -1$ $\quad c_3 = \underline{-1/2}$

Particular sol'n: $y = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-4t} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-6t}$

123 $y''' + 4y'' + y' - 6y = 0$

$y_1 = y$

$y_2 = y'$

$y_3 = y''$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 6y_1 - y_2 - 4y_3 \end{aligned}}$$

$$y''' = 6y - y' - 4y''$$
$$= 6y_1 - y_2 - 4y_3$$

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix} \mathbf{y}$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -1 & -4-\lambda \end{vmatrix} = -\lambda(-\lambda(-4-\lambda)+1) - 1(-6) = 0$$

$$-\lambda(4\lambda + \lambda^2 + 1) + 6 = 0$$

$$-\lambda^3 - 4\lambda^2 - \lambda + 6 = 0$$

$$\underline{\lambda^3 + 4\lambda^2 + \lambda - 6 = 0}$$

See that $\underline{\lambda = 1}$ is a solution.

$$(\lambda - 1) \cdot (\lambda^2 + 5\lambda + 6) = 0$$

$$(\lambda - 1)(\lambda + 2)(\lambda + 3) = 0$$

$$\lambda = \underline{1}, \ \lambda = \underline{-2}, \ \lambda = \underline{-3}$$

$$\underline{y} = c_1 \cdot \underline{v_1} \cdot e^t + c_2 \cdot \underline{v_2} \cdot e^{-2t} + c_3 \cdot \underline{v_3} e^{-3t}$$

$$\underline{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, \quad \underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{v_2} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \quad \underline{v_3} = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$$

12.4 $y = 3e^{-2t} - 5e^{t} + 12e^{-3t}$

a) Lin. second order : $y = y_h + y_p = C_1 e^{r_1 t} + C_2 e^{r_2 t} + y_p$

$y_h = 3e^{-2t} - 5e^{t} \to r_1 = -2,\ r_2 = 1$

$(r+2)(r-1) = 0$

$r^2 + r - 2 = 0$

Homogeneous eqn: $y'' + y' - 2y = 0$

$y_p = \underline{12e^{-3t}}$

$y = \underline{12e^{-3t}}$

$y' = 12e^{-3t} \cdot (-3)$

$= \underline{-36e^{-3t}}$

$y'' = -36\, e^{-3t} \cdot (-3)$

$= +\underline{108\, e^{-3t}}$

Diff. eqn: $y'' + y' - 2y = c(t)$

$108e^{-3t} + (-36)e^{-3t}$

$-2(12e^{-3t}) = c(t)$

$(108 - 36 - 24)e^{-3t} = c(t)$

$c(t) = \underline{48e^{-3t}}$

Diff. eqn:

$$\underline{\underline{y'' + y' - 2y = 48e^{-3t}}}$$

b) Linear third order:

$r_1 = -2 \quad r_2 = 1 \quad r_3 = -3$

Diff. eqn:

$y''' + 4y'' + y' - 6y = 0$

$(r+2)(r-1)(r+3)$

$= (r^2 + r - 2)(r+3)$

$= \underbrace{r^3 + 4r^2 + r - 6 = 0}_{\text{char. eqn.}}$

② Final exam 12/2021, Q2

a) $y'' + y' = 6e^{3t}$ $\quad y = y_h + y_p = \underline{\underline{C_1 + C_2 e^{-t} + \frac{1}{2}e^{3t}}}$

y_h: $y'' + y' = 0$

$r^2 + r = 0$
$r(r+1) = 0$
$r = 0,\ r = -1$

$\Bigg\}\ y_h = C_1 e^{0 \cdot t} + C_2 \cdot e^{-t}$
$= \underline{C_1 + C_2 e^{-t}}$

y_p: $y'' + y' = 6e^{3t}$ $\quad y = Ae^{3t}$
$y' = 3Ae^{3t}$
$y'' = 9Ae^{3t}$

$9Ae^{3t} + 3Ae^{3t} = 6e^{3t}$

$\underbrace{12A}_{=6} e^{3t} = 6e^{3t}$

$A = 1/2 \quad y_p = \frac{1}{2}e^{3t}$

b) $t(y' - y) = y \quad | : t$

$y' - y = \frac{y}{t}$

$y' = \frac{y}{t} + y = \underbrace{\left(\frac{1}{t} + 1\right)}_{f(t)} \underbrace{y}_{g(y)}$

$\frac{1}{y} y' = \frac{1}{t} + 1$

$\int \frac{1}{y}\, dy = \int \frac{1}{t} + 1\, dt$

$\ln|y| = \ln|t| + t + C$

$|y| = e^{\ln|t| + t + C} = |t| \cdot e^t \cdot e^C$

$y = \pm e^C t e^t = \underline{\underline{K t e^t}}$

Alt: $y' - (1 + \frac{1}{t})y = 0$

$\int -(1 + 1/t)\, dt = -t - \ln|t| + C$

$u = e^{-t - \ln|t|} = \frac{1}{e^t \cdot |t|}$

$\left(= \frac{1}{t \cdot e^t} \right)$

$(y \cdot u)' = 0$

$y \cdot u = C \quad y = \frac{C}{u}$
$= C \cdot e^t \cdot |t|$
$= K \cdot te^t$

Key Problems

Problem 1.
Solve the differential equations:

a) $y' = 3t^2 + 2$ b) $ty' = 1$ c) $y' = t\sqrt{t^2+1}$

Problem 2.
Solve the differential equations:

a) $y' = 5y$ b) $y' = y^2 t$ c) $y' = 5y(1 - y/10)$

Problem 3.
Solve the differential equations:

a) $y' + 3y = 6$ b) $y' - 2ty = 4t$ c) $y' + 2y = e^t$

Problem 4.
Solve the exact differential equations:

a) $3t^2 - 2t + 2y \cdot y' = 0$ b) $2y - 3t^2 + 2(y+t)y' = 0$ c) $\dfrac{y(1-2\ln t)}{t^3} + \dfrac{\ln t}{t^2} \cdot y' = 0$

Exercise problems

Problems from the textbook:	[E] 7.1 - 7.23
Exam problems:	[Final exam 11/2019] Q3ab

Answers to Key Problems

Problem 1.
a) $y = t^3 + 2t + C$ b) $y = \ln|t| + C$ c) $y = \frac{1}{3}(t^2+1)\sqrt{t^2+1} + C$

Problem 2.
a) $y = Ke^{5t}$ b) $y = -2/(t^2 + 2C)$ c) $y = 10 \cdot Ke^{5t}/(1 + Ke^{5t})$

Problem 3.
a) $y = 2 + Ce^{-3t}$ b) $y = -2 + Ce^{t^2}$ c) $y = \frac{1}{3}e^t + Ce^{-2t}$

Problem 4.
a) $y = \pm\sqrt{t^2 - t^3 + C}$ b) $y = -t \pm \sqrt{t^2 + t^3 + C}$ c) $y = \dfrac{Ct^2}{\ln t}$

Key Problems

Problem 1.
Find the equilibrium states and determine their stability. Sketch the solution curve $y = y(t)$.

a) $y' = 6 - 2y$ b) $y' = y^2 - 4$ c) $y' = 5y(1 - y/10)$

Problem 2.
Solve the differential equations:

a) $y'' + 6y' - 16y = 16t - 22$ b) $y'' + 6y' + 9y = 4e^{-t}$
c) $y'' - 3y' + 2y = 3e^{2t}$ d) $y'' - y = t^2$

Problem 3.
Show that there are separable, non-linear differential equations with the following functions as particular solutions:

a) $y = \sqrt{t^2 - 3}$ b) $y = \dfrac{2}{1 - t^2}$

Exercise Problems

Textbook problems:	[E] 7.24 - 7.34
Exam problems	[Final 12/2015] Q2, [Final 11/2017] Q2, [Final 11/2018] Q2ac, Q5, [Final 01/2021] Q3bc

Answers to Key Problems

Problem 1.
a) $y_e = 3$ is globally asymptotically stable
b) $y_e = -2$ is stable (but not globally asymptotically stable), $y_e = 2$ is unstable
c) $y_e = 0$ is unstable, $y_e = 10$ is stable (but not globally asymptotically stable)

Problem 2.
a) $y = C_1 e^{-8t} + C_2 e^{2t} + 1 - t$ b) $y = C_1 e^{-3t} + C_2 te^{-3t} + e^{-t}$
c) $y = C_1 e^{2t} + C_2 e^t + 3te^{2t}$ d) $y = C_1 e^t + C_2 e^{-t} - t^2 - 2$

Key Problems

Problem 1.
Write the systems of differential equations on matrix form and solve them:

a) $\begin{aligned} y_1' &= 2y_1 + 5y_2 \\ y_2' &= 5y_1 + 2y_2 \end{aligned}$ b) $\begin{aligned} y_1' &= y_2 \\ y_2' &= 4y_1 + 3y_2 \end{aligned}$ c) $\begin{aligned} y_1' &= y_1 + 4y_2 + 3 \\ y_2' &= y_1 - 2y_2 - 3 \end{aligned}$

Problem 2.
Solve the systems of differential equations:

a) $\mathbf{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ b) $\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$

Problem 3.
Rewrite the differential equation $y''' + 4y'' + y' - 6y = 0$ as a system of first order linear differential equations, and solve the system of differential equations.

Problem 4.
Let $y(t) = 3e^{-2t} - 5e^{t} + 12e^{-3t}$.

a) Find a linear second order differential equation that has y as a particular solution.
b) Find a linear third order differential equation that has y as a particular solution.
c) Find a 3×3 matrix A such that $\mathbf{y}' = A\mathbf{y}$ has $\mathbf{y} = (y, y', y'')$ as a particular solution.

Problem 5.
Find the equilibrium states and determine their stability:

a) $y'' + 7y' + 10y = 5$ b) $y'' + y' - 20y = 1$ c) $y''' + 4y'' + y' - 6y = 12$

d) $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ e) $\mathbf{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \mathbf{y}$

Exercise Problems

Problems from the textbook	[E] 9.1 - 9.7
Final exam problems	11/2018 Q2,Q5, 01/2019 Q2, 01/2020 Q3, 03/2021 Q3bc

Eivind Eriksen, Office B3y-085, eivind.eriksen@bi.no
https://www.dr-eriksen.no/teaching/GRA6035/
https://www.dr-eriksen.no/teaching/ELE3781/
BI Norwegian Business School
MSc Business/Finance
BSc Data Science for Business

Exam	Final exam in GRA 6035 Mathematics
Date	December 2nd, 2021 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 2 \\ -3 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A, and find a base of the column space of A.
(b) **(6p)** Show that $\mathbf{v}$ is an eigenvector of A, and find the corresponding eigenvalue.
(c) **(6p)** Find the determinant of A.

Let S be a symmetric 3×3 matrix with eigenvalues $\lambda = 1$, $\lambda = 2$ and $\lambda = 4$.

(d) **(6p)** Show that S has an inverse matrix, and determine the definiteness of S^{-1}.

Question 2.

(a) **(6p)** Solve the differential equation $y'' + y' = 6e^{3t}$.
(b) **(6p)** Solve the differential equation $t(y' - y) = y$.
(c) **(6p)** Solve the difference equation $y_{t+2} + 3y_{t+1} - 4y_t = 5$.
(d) **(6p)** Solve the system of differential equations:

$$\mathbf{y}' = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix}$$

Question 3.
Let $g(x, y, z, w) = 3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 7z^2 + 4w^2$, and consider the Kuhn-Tucker problem given by

$$\max f(x, y, z) = x + y + z + w \text{ subject to } g(x, y, z, w) \leq 18$$

(a) **(6p)** Determine the definiteness of the quadratic form g.
(b) **(6p)** Write down the Kuhn-Tucker conditions of the problem in matrix form.
(c) **(6p)** Write down the non-degenerate constraint qualification in this problem, and find all admissible points where this condition does not hold (if there are any).
(d) **(6p)** Solve the Kuhn-Tucker problem.
(e) **Extra credit (6p)** Determine whether the set $D = \{(x, y, z, w) : g(x, y, z, w) \leq 18\}$ of admissible points is a compact set.