Problem Set 13: Solution to Key Problems

1. a) $y_{t+1} = 1.04 y_t \; , \; y_0 = 100$

$y_{t+1} - 1.04 y_t = 0$

$r - 1.04 = 0 \quad r = 1.04 \quad y_t = \underline{C \cdot 1.04^t}$

$y_0 = C \cdot 1.04^0 = C \cdot 1 = C = 100$

$\Downarrow$

$y_t = \underline{\underline{100 \cdot 1.04^t}}$

b) $y_{t+1} - 6 y_t = 10t + 3$

$y_t = y_t^h + y_t^p = \underline{\underline{C \cdot 6^t - 2t - 1}}$

$\underline{y_t^h}$: $y_{t+1} - 6 y_t = 0$

$r - 6 = 0$

$r = 6 \quad \Rightarrow y_t^h = \underline{C \cdot 6^t}$

$\underline{y_t^p}$: $y_t = At + B$

$y_{t+1} = A(t+1) + B = At + A + B$

$\left.\right\} \; (At + A + B) - 6(At + B) = 10t + 3$

$\underbrace{(-5A)}_{=10} t + \underbrace{(A - 5B)}_{=3} = 10t + 3$

$y_t^p = \underline{-2t - 1} \Leftarrow \quad \underline{A = -2} \quad -2 - 5B = 3$

$-5B = 5$

$\underline{B = -1}$

c) $y_{t+2} - 3y_{t+1} + 2y_t = 0$

$y_t = y_t^h = \underline{\underline{C_1 + C_2 \cdot 2^t}}$

y_t^h: $r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$\underline{r = 1}, \; \underline{r = 2} \quad \Rightarrow y_t^h = C_1 1^t + C_2 \cdot 2^t = \underline{C_1 + C_2 \cdot 2^t}$

d) $y_{t+2} - 5y_{t+1} + 6y_t = 2t$

$y_t = y_t^h + y_t^p = \underline{\underline{C_1 \cdot 2^t + C_2 \cdot 3^t + t + 3/2}}$

$\underline{y_t^h}$: $y_{t+2} - 5y_{t+1} + 6y_t = 0$

$r^2 - 5r + 6 = 0$

$r = 2, \; r = 3 \quad \Rightarrow y_t^h = \underline{C_1 \cdot 2^t + C_2 \cdot 3^t}$

y_t^p: $y_t = At+B$
$y_{t+1} = A(t+1)+B = At+A+B$
$y_{t+2} = A(t+2)+B = At+2A+B$

$$\left.\begin{array}{l} y_t = At+B \\ y_{t+1} = A(t+1)+B = At+A+B \\ y_{t+2} = A(t+2)+B = At+2A+B \end{array}\right\} \begin{array}{l}(At+2A+B)-5(At+A+B) \\ \quad +6(At+B) = 2t \end{array}$$

$(A-5A+6A)t + (2A+B-5A-5B+6B) = 2t$

$\underbrace{(2A)}_{=2}t + \underbrace{(-3A+2B)}_{=0} = 2t$

$\underline{A=1}$ $\qquad -3+2B=0$
$\underline{B=3/2}$

$\underline{y_t^p = t+3/2}$

e) $y_{t+2}-4y_{t+1}+4y_t = 1$

$y_t = y_t^h + y_t^p = \underline{\underline{(C_1+C_2t)\cdot 2^t+1}}$

$\underline{y_t^h}$: $r^2-4r+4=0$
$r_1=r_2=2$ $\Rightarrow$ $\underline{y_t^h = (C_1+C_2t)\cdot 2^t}$

$\underline{y_t^p}$: $y_t = A$: $A-4A+4A=1$
$A=1$ $\Rightarrow$ $y_t^p = \underline{1}$

f) $y_{t+2}+y_{t+1}-2y_t = 6$

$y_t = y_t^h + y_t^p = \underline{\underline{C_1\cdot(-2)^t + C_2 + 2t}}$

$\underline{y_t^h}$: $r^2+r-2=0$
$(r+2)(r-1)=0$
$r=-2, r=1$ $\Rightarrow$ $y_t^h = C_1\cdot(-2)^t + C_2\cdot 1^t = \underline{C_1(-2)^t + C_2}$

$\underline{y_t^p}$: $y_t = A$ $\qquad A+A-2A=6$
$0\cdot A = 6$ impossible $\to$ multiply $\underline{\text{with } t}$

$$\left.\begin{array}{l} y_t = At \\ y_{t+1} = A(t+1) = At+A \\ y_{t+2} = A(t+2) = At+2A \end{array}\right\} \begin{array}{l}(At+2A)+(At+A)-2(At)=6 \\ 0\cdot At + 3A = 6 \\ 3A=6 \\ A=2 \end{array}$$

$\underline{y_t^p = 2t}$

$\underline{2.}$ In each case, $y_t = \begin{pmatrix} y_t \\ z_t \end{pmatrix}$ gives the system
$y_{t+1} = A\cdot y_t + \underline{b}$ for a 2x2 matrix A and a 2-vector $\underline{b}$.
We use eigenvalues / eigenvectors of A computed in Key Problem 12.1 a) – c):

a) $\underline{\underline{y_t = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 7^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot (-3)^t}}$

b) $\underline{\underline{y_t = c_1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot 4^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot (-1)^t}}$

c) Constant solution: $\underline{y}_{t+1} = \underline{y}_t \iff A\underline{y}_t + \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \underline{y}_t$

$A\underline{y}_t - \underline{y}_t = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$(A - I)\underline{y}_t = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 0 & 4 \\ 1 & -3 \end{pmatrix} \underline{y}_t = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$4z_t = -3$

$y_t - 3z_t = 3$

$\left\{ \begin{array}{l} z_t = -3/4 \\ y_t = 3 + 3 \cdot (-3/4) \\ \quad = \frac{12}{4} - \frac{9}{4} = \frac{3}{4} \end{array} \right. \implies \underline{y}_t^p = \underline{\begin{pmatrix} 3/4 \\ -3/4 \end{pmatrix}}$

$\underline{\underline{\underline{y}_t = c_1 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot 2^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot (-3)^t + \begin{pmatrix} 3/4 \\ -3/4 \end{pmatrix}}}} \impliedby \underline{y}_t^p$

3. a) $p_{t+2} - 2p_{t+1} + p_t = -15, \quad p_0 = 695, \quad p_1 = 743$

Alt 1: $p_t = p_t^h + p_t^p = \underline{c_1 + c_2 t - 7.5t^2} = \underline{\underline{695 + 55.5t - 7.5t^2}}$

p_t^h: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_1 = r_2 = 1$

$p_t^h = c_1 \cdot 1^t + c_2 \cdot t \cdot 1^t = \underline{c_1 + c_2 t}$

p_t^p:

$\left. \begin{array}{l} p_t = C \\ p_{t+1} = C \\ p_{t+2} = C \end{array} \right\}$ $C - 2C + C = -15$

$0 \cdot C = -15$

no part. sol'n.

$\left. \begin{array}{l} p_t = Ct \\ p_{t+1} = C(t+1) \\ p_{t+2} = C(t+2) \end{array} \right\}$ $C(t+2) - 2C(t+1) + Ct = -15$

$0 \cdot t + 0 = -15$

no part. sol'n.

$$\left.\begin{aligned} p_t &= ct^2 \\ p_{t+1} &= c(t+1)^2 \\ p_{t+2} &= c(t+2)^2 \end{aligned}\right\}$$

$$c(t^2+4t+4) - 2c(t^2+2t+1) + ct^2 = -15$$

$$0\cdot t^2 + 0\cdot t + (4c-2c) = -15$$

$$2c = -15 \qquad c = -15/2$$

$$p_t^p = -\frac{15}{2}t^2 = \underline{-7.5t^2}$$

Initial conditions:

$p_0 = 695$: $\quad p(0) = c_1 + c_2\cdot 0 - 7.5\cdot 0^2 = 695 \quad c_1 = \underline{695}$

$p_1 = 743$: $\quad p(1) = 695 + c_2\cdot 1 - 7.5\cdot 1^2 = 743$

$$c_2 = 743 - 695 + 7.5 = \underline{55.5}$$

Alt 2: Use hint from (b):

$$d_t = p_{t+1} - p_t$$

$$\Rightarrow d_{t+1} - d_t = (p_{t+2} - p_{t+1}) - (p_{t+1} - p_t)$$

$$= p_{t+2} - 2p_{t+1} + p_t = -15$$

Difference eqn: $\quad d_{t+1} - d_t = -15, \quad d_0 = p_1 - p_0 = 48$

$d_{t+1} - d_t = -15, \quad d_0 = 48$:

$$d_t = d_t^h + d_t^p = \underline{c - 15t}$$

$$d_0 = c - 15\cdot 0 = 48$$
$$c = 48$$

$$d_t = \underline{\underline{48 - 15t}}$$

d_t^h: $\quad r - 1 = 0$
$\quad r = 1 \quad \Rightarrow \quad d_t^h = c\cdot 1^t = \underline{c}$

d_t^p: $\quad d_{t+1} - d_t = -15$

$d_t = A$: $\quad A - A = -15$ impossible

$$\left.\begin{aligned} d_t &= At \\ d_{t+1} &: A(t+1) = At + A \end{aligned}\right) \quad (At + A) - At = -15 \qquad d_t^p = \underline{-15t}$$
$$A = -15$$

$$d_t = p_{t+1} - p_t = 48 - 15t$$

$$p_t = p_t^h + p_t^p = \underline{c - 7.5t^2 + 55.5t}$$

p_t^h: $\quad r - 1 = 0$
$\quad r = 1 \qquad p_t^h = c\cdot 1^t = c$

P_t^P:

$P_t = At + B$

$P_{t+1} = A(t+1) + B$
$= At + A + B$

$(At + A + B) - (At + B) = 48 - 15t$

$0 \cdot t + A = 48 - 15t$

impossible

$p_t = (At+B)t = At^2 + Bt$

$P_{t+1} = A(t+1)^2 + B(t+1)$
$= At^2 + 2At + A + Bt + B$
$= At^2 + (2A+B)t + (A+B)$

$P_{t+1} - P_t = At^2 + (2A+B)t + (A+B) - (At^2 + Bt)$
$= (2A)t + (A+B) = 48 - 15t$

$2A = -15 \Rightarrow A = -15/2 = -7.5$

$A + B = 48 \Rightarrow B = 48 - (-15/2)$
$= 48 + 7.5$
$= 55.5$

$p_0 = 695$: $\quad C - 7.5 \cdot 0^2 - 55.5 \cdot 0 = 695$
$c = 695$

$$P_t = 695 + 55.5t - 7.5t^2$$

b) $d_{t+1} - d_t = -15$ (see above)

$d_t = 48 - 15t \Rightarrow d_t > 0$ for $t < \frac{48}{15} = 3.2$, that is for $t = 0, 1, 2, 3$

$d_t < 0$ for $t \geq 4$

4. a) $A = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix}$

Eigenvalues:

$(-3-\lambda)(\lambda^2 + 10\lambda + 24) = 0$

$\lambda = -3, \lambda = -4, \lambda = -6$

Eigenvectors:

$\lambda = -3$: $\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = -4$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = -6$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\underline{y}_t = c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot (-3)^t + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot (-4)^t + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot (-6)^t$$

$$y_0 = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \underset{\text{use Gauss}}{\Rightarrow} \begin{array}{l} c_1 = 0 \\ c_2 = 1/2 \\ c_3 = -1/2 \end{array}$$

$$\underline{\underline{y_t = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot (-4)^t - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot (-6)^t}}$$

b) $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & 0 \\ 3 & -1 & 1-\lambda \end{vmatrix} = (2-\lambda) \cdot \left[(2-\lambda)(1-\lambda) - 3 \right] - (-1) \left[(1-\lambda) + 1 \right]$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2 - 3) + (2-\lambda)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda - 1 + 1)$$

$$= (2-\lambda)(\lambda^2 - 3\lambda) = 0$$

$$(2-\lambda) \cdot \lambda \cdot (\lambda - 3) = 0$$

$$\underline{\lambda = 2}, \quad \underline{\lambda = 0}, \quad \underline{\lambda = 3}$$

Eigenvectors:

$\lambda_1 = 2$: $\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 3 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 0$: $\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 2 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$

$\lambda_3 = 3$: $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 3 & -1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \underline{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\underline{y}_t = c_1 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot 2^t + c_2 \cdot \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} \cdot 0^t + c_3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot 3^t$$

$$\underline{y}_0 = c_1 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \underset{\underline{\text{Gauss}}}{\Rightarrow} \begin{matrix} c_1 = 1 \\ c_2 = 0 \\ c_3 = 1 \end{matrix}$$

$$\underline{y}_t = \underline{\underline{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot 2^t + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot 3^t}}$$