Plan

1 Linear difference equations
2 Systems of linear first order difference equations
3 Equilibrium states and stability

1 Difference equations

Defn. A difference equation is an equation where the unknown is a sequence, $(y_t)_{t=0,1,2,\dots} = y_0, y_1, y_2, \dots$ and the equation contains y_t, y_{t+1} (and possibly y_{t+2})

Discrete time

Sequences instead of continuous functions $y=y(t)$

change:

$\Delta(y_t) = y_{t+1} - y_t$

Ex: $y_{t+1} = 1.04\, y_t$, $y_0 = 100$

$y_1 = 1.04\, y_0$

$y_2 = 1.04\, y_1 = 1.04^2\, y_0$

$\vdots$

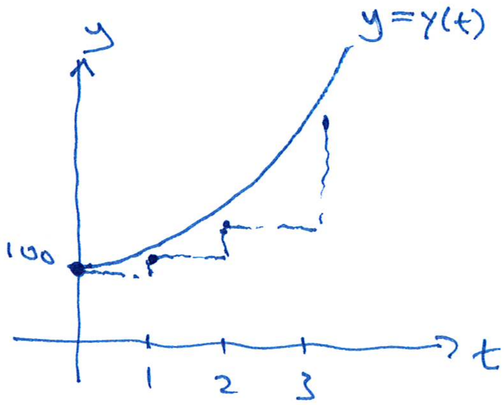
$y_t = \underline{1.04^t \cdot y_0}$

$\Rightarrow y_t = \underline{\underline{100 \cdot 1.04^t}}$

Differential equation

$y' = 0.04\, y$, $y_0 = 100$

Sol: $y = \underline{\underline{100 \cdot e^{0.04t}}}$



Note: $y_{t+1} = 1.04 \cdot y_t$

$y_{t+1} - y_t = 1.04\, y_t - y_t$

$y_{t+1} - y_t = 0.04\, y_t$

Linear first order difference equations

$$y_{t+1} + a \cdot y_t = b_t \quad \begin{cases} a: \text{constant} \\ b_t: \text{expr. in } t \end{cases}$$

First order diff. eqn:

$$y_{t+1} = F(t, y_t)$$

General solution: one undetermined coeff.

Method: Superposition $y_t = y_t^h + y_t^p$

General solution of the hom. eqn $y_{t+1} + a y_t = 0$

Particular sol'n of $y_{t+1} + a y_t = b_t$

Ex: $y_{t+1} - 1.10\, y_t = -300, \; y_0 = 10.000$

y_t^h: $y_{t+1} - 1.10\, y_t = 0 \longrightarrow y_{t+1} = 1.10\, y_t$

$y_t = \underline{C \cdot 1.10^t}$

Char. eqn: $r - 1.10 = 0$

$y_{t+1} \rightsquigarrow r$ $\quad r = \underline{1.10}$

$y_t \sim 1$

$y_t^h = C \cdot 1.10^t$

y_t^p: $y_{t+1} - 1.10\, y_t = -300$ Method of undetermined coeff.

$$\left.\begin{aligned} y_t &= A \\ y_{t+1} &= A \end{aligned}\right\} \quad \begin{aligned} A - 1.10 A &= -300 \\ -0.10 A &= -300 \\ A = \frac{-300}{-0.10} &= 3000 \end{aligned} \qquad y_t^p = \underline{3000}$$

Interpretation:

$y_{t+1} - 1.10\, y_t = -300, \; y_0 = 10'$

$y_{t+1} - y_t = 0.10\, y_t - 300$

Bank acc., 10% interest per time period, withdr. 300 at end of each time period

General solution:

$$y_t = y_t^h + y_t^p = \underline{C \cdot 1.10^t + 3000}$$

$y_0 = \underline{10.000}$: $10.000 = C \cdot 1.10^0 + 3000$

$C = 7000$

$$10.000 \cdot 1.10^t - 3000 \cdot (1.10^t - 1) = $$

Particular solution: $y_t = \underline{7000 \cdot 1.10^t + 3000}$

Second order linear difference equations:

$$y_{t+2} + a y_{t+1} + b y_t = c_t \quad \left\{ \begin{array}{l} a, b \text{ const.} \\ c_t : \text{expr. in } t \end{array} \right.$$

Ex: $y_{t+2} - 5y_{t+1} + 6y_t = 2t+1$

Superposition: $y_t = y_t^h + y_t^p = C_1 \cdot 2^t + C_2 \cdot 3^t + t+2$

y_t^h: $y_{t+2} - 5y_{t+1} + 6y_t = 0$

Char. eqn.: $r^2 - 5r + 6 = 0 \rightarrow y_t^h = C_1 \cdot 2^t + C_2 \cdot 3^t$

$y_{t+2} \rightsquigarrow r^2$ $\quad r=2, \; r=3$

$y_{t+1} \rightsquigarrow r$

$y_t \rightsquigarrow 1$

y_t^p: $y_{t+2} - 5y_{t+1} + 6y_t = 2t+1$

$y_t = At+B$

$y_{t+1} = A(t+1) + B$
$= At + A + B$

$y_{t+2} = A(t+2) + B$
$= At + 2A + B$

$(At + 2A + B) - 5(At + A + B) + 6(At+B) = 2t+1$

$2At + (-3A + 2B) = 2t+1$

$2A = 2 \qquad -3A + 2B = 1$

$A = 1 \qquad -3 + 2B = 1$

$B = 2 \qquad y_t^p = t+2$

Second order difference equation:

Contains y_{t+2}, y_{t+1}, y_t, has two undetermined coeff. C_1, C_2.

$\Delta(y_t) = y_{t+1} - y_t$

$\Delta^2(y_t) = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) = \underline{y_{t+2} - 2y_{t+1} + y_t}$

(y_t) :	0	1	$2^2=4$	9	16	25	$y_t = t^2$
$\Delta(y_t)$:	1	3	5	7	9		$2t+1$
$\Delta^2(y_t)$:	2	2	2	2			

Ex: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 1$, $y_1 = 1$ Fibonacci seq.

1 1 2 3 5 8 13 ...

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(2) Systems of difference equations:

$y_{t+1} = 2y_t - z_t + 3$

$z_{t+1} = -y_t + 2z_t - 3$

$$\begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$\underline{y}_t = \begin{pmatrix} y_t \\ z_t \end{pmatrix}$: $\quad \underline{y}_{t+1} = A \cdot \underline{y}_t + \underline{b} \iff \underline{y}_{t+1} - A\underline{y}_t = \underline{b}$

Superposition: $\underline{y}_t = \underline{y}_t^h + \underline{y}_t^p$

Ex: $\underline{y}_{t+1} = A\underline{y}_t + \underline{b} \qquad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$\underline{y}_t^h$: $\quad \underline{y}_{t+1} = A \cdot \underline{y}_t \Rightarrow \underline{y}_t^h = c_1 \cdot \underline{v}_1 \cdot \lambda_1^t + c_2 \cdot \underline{v}_2 \lambda_2^t$

$$= \underline{c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t}$$

Eigenvalues:

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = \underline{1}, \lambda_2 = \underline{3}$

Eigenvectors:

E_1: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad x - y = 0$, y free $\qquad E_3$: $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ $\quad -x - y = 0$, y free

$\begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{y}_t^p$: $\quad \underline{y}_{t+1} - A\underline{y}_t = \underline{b} \qquad$ Note: $\underline{y}_t^p \stackrel{?}{=} \underline{y}_e$

$\underline{y}_t = \begin{pmatrix} A \\ B \end{pmatrix} \quad \begin{pmatrix} A \\ B \end{pmatrix} - A \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \underline{b}$

$\underline{y}_e - A\underline{y}_e = \underline{b}$

$\underline{y}_e - \underline{b} = A\underline{y}_e$

$A\underline{y}_e = \underline{y}_e - \underline{b}$

$A\underline{y}_e - \underline{y}_e = -\underline{b}$

$$\boxed{(A - I)\,\underline{y}_e = -\underline{b}}$$

Lin sys to find $\underline{y}_t^p = \underline{y}_e$ in this type of systems.

Ex: $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $A - I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $-\underline{b} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & -1 & -3 \\ -1 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 0 & 0 \end{array}\right)$ $y - z = -3$ $y = \underline{z-3}$

z free

Choose $\underline{y}_t^p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ (one of the sol'n, with $z=0$)

General solution: $\underline{y}_t = \underbrace{c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t}_{\underline{y}_t^h} + \underbrace{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}_{\underline{y}_t^p}$

3 Equilibrium states and stability

Autonomous difference equations (no explicit t)

i) $y_{t+1} + a y_t = b$:

$y_t = y_e$ const: $y_e + a \cdot y_e = b$

$(1+a) y_e = b$

$y_e = \dfrac{b}{1+a}$ $(a \neq -1)$

Eq. states = constant sol'ns

General sol: $y_t = C \cdot (-a)^t + y_e$ Stable if $|a| < 1$ $(|-a| < 1)$

Ex: $y_{t+1} - 1.10 y_t = -300$ $y_e = \dfrac{-300}{-0.10} = \underline{3000} = y_t^p$

$y_t = C \cdot 1.10^t + 3000$ unstable since $|a| = 1.10$

$\downarrow$

∞

(as $t \to \infty$)

ii) $y_{t+2} + a y_{t+1} + b y_t = c$

y_e: $y_e + a y_e + b y_e = c$

$$y_e = \frac{c}{1+a+b}$$

$(1+a+b \neq 0)$

Stable if $|r_1|, |r_2| < 1$

$y_t^h = C_1 r_1^t + C_2 r_2^t$

$y_t^p = y_e$

iii) $\underline{y}_{t+1} = A \underline{y}_t + \underline{b}$

$\underline{y_e}$: Solution of lin. sys.: $(A - I)\underline{y_e} = -\underline{b}$

Stable: if $|\lambda_i| < 1$ for all i

$\underline{y_t}^h = C_1 \cdot \underline{v_1} \lambda_1^t + C_2 \underline{v_2} \lambda_2^t + \ldots + C_n \underline{v_n} \lambda_n^t$

$\underline{y_t}^p = \underline{y_e}$

Ex: Markov chain (regular)

$$\underline{v}_{t+1} = \begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix} \underline{v_t}, \qquad \underline{v_t} = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix} \cdot \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$\underline{v}_{t+1} = A \cdot \underline{v_t}$

E_1: $\begin{pmatrix} -0.15 & 0.25 \\ \cancel{0.15} & \cancel{-0.25} \end{pmatrix}$

$-0.15x + 0.25y = 0$

y free

$$x = \frac{0.25y \cdot 20}{0.15 \cdot 20} = \frac{5y}{3}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5y/3 \\ y \end{pmatrix} = \frac{y}{3}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

State vector: $5+3=8$

$$\Rightarrow \underline{v} = \frac{1}{8}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/8 \\ 3/8 \end{pmatrix} \leftarrow \text{Eq. state globally asymp. stable}$$

$$\underline{v_t} = C_1 \cdot \underline{v_1} \overset{1}{\lambda_1^t} + C_2 \underline{v_2} \overset{0.6}{\lambda_2^t}$$

$$= C_1 \begin{pmatrix} 5 \\ 3 \end{pmatrix} + C_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} 0.6^t = \underline{\frac{1}{8}\begin{pmatrix} 5 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \overset{\to 0}{0.6^t}} \to \begin{pmatrix} 5/8 \\ 3/8 \end{pmatrix} = \underline{v} \quad \text{as } t \to \infty$$

Ex: $y_{t+2} = y_{t+1} + y_t$, $y_0 = 1$, $y_1 = 1$ 1 1 2 3 5 8 ...

$y_{t+2} - y_{t+1} - y_t = 0$ homog.

Char. equ. $r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$

$y_0 = 1$: $C_1 + C_2 = 1$ $C_2 = 1 - C_1$

$y_1 = 1$: $C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + C_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$ $| \cdot 2$

$C_1(1+\sqrt{5}) + C_2(1-\sqrt{5}) = 2$

$C_1(1+\sqrt{5}) + (1-C_1)(1-\sqrt{5}) = 2$

$\cancel{C_1} + C_1\sqrt{5} + 1 - \cancel{C_1} - \sqrt{5} + C_1\sqrt{5} = 2$

$C_1 \cdot 2\sqrt{5} = 1 + \sqrt{5}$

$C_1 = \frac{1}{\sqrt{5}} \frac{1+\sqrt{5}}{2}$

$C_2 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5} - (1+\sqrt{5})}{2\sqrt{5}} = \frac{-1+\sqrt{5}}{2\sqrt{5}} = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$

$$y_t = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{t+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{t+1}$$