Plan

1 About the exam
2 Final exam 11/2023

Remember:

Course evaluation!

② Final Exam 11/2023

1. (a) $Y_{t+2} - Y_{t+1} - 2Y_t = 0$

Homog. second order diff. equ.

Char. equ. $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \quad r = 2,\ r = -1$

$y_t = C_1 \cdot 2^t + C_2 \cdot (-1)^t$

(b) $\underline{Y}_{t+1} = A\underline{y}_t \qquad A = \begin{pmatrix} 0.94 & 0.14 \\ 0.06 & 0.86 \end{pmatrix}$ regular Markov chain since $A > 0$

$\underline{\lambda = 1}$: E_1 : $\begin{pmatrix} -0.06 & 0.14 \\ \cancel{0.06} & -0.14 \end{pmatrix}$ $\qquad -0.06x + 0.14y = 0 \qquad x = \frac{-0.14y}{-0.06} = \frac{14}{6}y = \frac{7}{3}y$

y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7/3 \cdot y \\ y \end{pmatrix} = \frac{y}{3}\begin{pmatrix} 7 \\ 3 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ base of E_1

$7 + 3 = 10 \Rightarrow \underline{v} = \frac{1}{10}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

Eq. state.

(c) $\underline{v}_4 = x\underline{v}_1 + y\underline{v}_2 + z \cdot \underline{v}_3$

$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & 7 & -4 \\ 3 & 4 & 10 & -3 \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -6 \\ \cancel{0} & \cancel{1} & \cancel{1} & \cancel{-6} \end{array}\right) \xrightarrow{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x \quad y \quad z$

There are inf. many ways to write $\underline{v}_4$ as a lin. comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$. $\Longleftarrow$ One degree of freedom

(d) $\min f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 5 \end{pmatrix} \quad \begin{matrix} D_1 = 1 \\ D_2 = 2 \\ D_3 = 2 \cdot 1 \end{matrix}$$

$B = (2 \;\; 0 \;\; -6)$

$C = 5$

Stat. pts.: $f'(\underline{x}) = 2A\underline{x} + B^T = \underline{0}$

$H(f) = 2A$

A pos. defn.

$2A$ — " — $\Rightarrow f$ convex

$2A\underline{x} = -B^T \quad | : 2$

$A\underline{x} = -\frac{1}{2}B^T$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 5 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$x - 2z = -1 \quad \underline{x = 1}$

$2y = 0 \quad \underline{y = 0}$

$\underline{z = 1}$

$\Rightarrow (1,0,1)$ glob min pt of f $\Rightarrow f_{min} = f(1,0,1) = 1 + 0 + 5 - 4 + 2 - 6 + 5$

$= \underline{\underline{3}}$

2. a) $$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \\ 1 & 18 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 \\ 0 & -7 & 1 \\ 0 & 14 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 \\ 0 & -7 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rk}\, A = \underline{\underline{2}} \qquad \det(A) = 1 \cdot (-7) \cdot 0 = \underline{\underline{0}}$

b) Null(A): $A\underline{x} = \underline{0}$ $\qquad (A \,|\, \underline{0}) \to \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -7 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$-7y + z = 0 \qquad -7y = -z$ $\qquad z$ free

$y = \underline{z/7}$

$x + 4(z/7) + 2z = 0$

$x = -\frac{4}{7}z - \frac{14}{7}z = \underline{-18z/7}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -18z/7 \\ z/7 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -18/7 \\ 1/7 \\ 1 \end{pmatrix} = \frac{z}{7}\begin{pmatrix} -18 \\ 1 \\ 7 \end{pmatrix}$$

Base of Null(A): $\underline{v} = \underline{\underline{\begin{pmatrix} -18 \\ 1 \\ 7 \end{pmatrix}}}$

(c) From (a) or (b) we know that $\lambda = 0$ is an eigenvalue of A

Note: $\text{Null}(A) = E_0$

$A\underline{x} = \underline{0}$ $\quad A\underline{x} = 0 \cdot \underline{x}$

If λ is an eigenvalue of A:

$1 \leq \dim E_\lambda \leq \text{mult.}\,\lambda$

Char. eqn: $-\lambda^3 + \text{tr}(A)\lambda^2 - c_2\lambda + \underbrace{\det(A)}_{=0} = 0$

$\text{tr}(A) = 2$

$c_2 = -7 - 90 - 2 = -99$

$\det(A) = 0$

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \\ 1 & 18 & 0 \end{pmatrix}$$

$$\underline{\underline{-\lambda^3 + 2\lambda^2 + 99\lambda = 0}}$$

Eigen val:

$-\lambda(\lambda^2 - 2\lambda - 99) = 0$

$\underline{\underline{\lambda = 0}}$ or $\lambda^2 - 2\lambda - 99 = 0$

$(\lambda - 11)(\lambda + 9) = 0$

$\underline{\underline{\lambda = 11}}, \ \underline{\underline{\lambda = -9}}$

Alt: $\begin{vmatrix} 1-\lambda & 4 & 2 \\ 2 & 1-\lambda & 5 \\ 1 & 18 & -\lambda \end{vmatrix} = 0 \Rightarrow \dots \Rightarrow \underline{-\lambda^3 + 2\lambda^2 + 99\lambda = 0}$

Concl: $\underbrace{\underline{\underline{-\lambda^3 + 2\lambda^2 + 99\lambda = 0}}}_{\text{char. eqn.}}$, $\underbrace{\underline{\underline{\lambda = 0, \lambda = 11, \lambda = -9}}}_{\text{eigenval. of } A.}$

(d) We claim: If λ is an eigenvalue of A with eigenvector $\underline{v}$, then λ^2 is an eigenval. of A^2 with eigenvector $\underline{v}$

since

$A \cdot |\ A\underline{v} = \lambda\underline{v}$

$A^2\underline{v} = A \cdot (\lambda\underline{v})$

$= \lambda(A\underline{v})$

$= \lambda(\lambda\underline{v})$

$= \lambda^2\underline{v}$

$\Rightarrow$ Eigenvalues of $B = A^2$: $0^2 = 0$, $11^2 = 121$, $(-9)^2 = 81$

$\lambda_B = \underline{\underline{0, 121, 81}}$

Dimension of Null(B)

$\text{Null}(B) = E_0(B)$ 1-dimensional

$\dim \text{Null}(B) = \underline{\underline{1}}$

3. max $f(\underline{x}) = 27 + \underline{x}^T A \underline{x}$ when $\underline{x}^T D \underline{x} = 10$

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

a) $H(f) = 2A$

A neg. defn. $\Leftrightarrow$ 2A neg. defn. $\Leftrightarrow$ f concave

Definiteness of A:

$D_1 = -1$

$D_2 = 2$

$D_3 = -2(2-1) = -2$

$$D_4 = +1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{vmatrix} + (-6) \cdot D_3$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \end{pmatrix}$$

$= +1(-1 \cdot (+2-1)) + (-6) \cdot (-2)$

$= \cancel{8}\,{-1} + 12 = \cancel{20}\ \underline{11}$

b) $\boxed{\begin{array}{l} \text{FOC: } L'(\underline{x}) = 2A\underline{x} - \lambda(2D\underline{x}) = \underline{0} \\ \text{C: } \underline{x}^T D \underline{x} = 10 \end{array}}$

$L = 27 + \underline{x}^T A \underline{x} - \lambda(\underline{x}^T D \underline{x} - 10)$

Substitute $\lambda = -2$, and solve FOC+C to find cand. pts.

$2A\underline{x} + 4D\underline{x} = \underline{0} \quad | :2$

$A\underline{x} + 2D\underline{x} = 0$

$(A+2D)\underline{x} = \underline{0}$

$$A + 2D = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -5 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -5 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 3 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{-3} \end{pmatrix}$$

$$\begin{aligned} -x \quad\quad + z + w &= 0 \\ -y \quad + 2w &= 0 \\ -z + 3w &= 0 \end{aligned}$$

w free

$\cancel{x = 2w}$ $\cancel{y = 2w}$ $\cancel{z = 2w}$

$x = 4w$, $y = 2w$, $z = 3w$

FOC: $(x,y,z,w) = (4w, 2w, 3w, w)$ w free

C: $xw + yz = 4w^2 + 6w^2 = 10w^2 = 10$ $\quad w^2 = +1$, $w = \pm 1$

Candidate pts: $(4,2,3,1;-2)$, $(-4,-2,-3,-1;-2)$

(c) test cand pts with $\lambda = -2$ using SOC:

$h(\underline{x}) = \mathcal{L}(\underline{x};-2) = 27 + \underline{x}^T A \underline{x} + 2(\underline{x}^T D \underline{x} - 10)$

$H(h) = 2A + 2(2D) = 2(A + 2D)$

$$A + 2D = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -6 \end{pmatrix}$$

$D_1 = -1$
$D_2 = 2$
$D_3 = -1 \cdot 3 + 1 \cdot 2 = -1$
$D_4 = 0$ from (b) $\Rightarrow H(h)$ neg. semidef. by the RRC.

$\Rightarrow h$ concave

$\overset{SOC}{\Rightarrow}$ $(4,2,3,1), (-4,-2,-3,-1)$ are max. pts with $\lambda = -2$

$f_{max} = f(4,2,3,1) = 7$ max value

d) $xw - yz = 10$: D compact $\Leftrightarrow \begin{cases} \text{i) closed ok since the C is an equation} \\ \text{ii) bounded} \end{cases}$

$x = 100$: $w = \frac{1}{10}$ $y = z = 0$

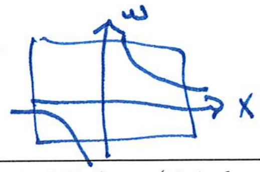
$x = a$: $w = \frac{10}{a}$, $y = z = 0$

$(a, 0, 0, \frac{10}{a})$ is adm for all $a > 0$

D is not bounded

$\Rightarrow$ D is not compact

Alt:



$xw = 10$
$w = 10/x$
$(y = z = 0)$

4. a) $y' + 4ty = 8t \qquad y' = 8t - 4ty$

$y' = 4t \cdot (2-y) \quad \text{sep.} \ | : (2-y)$

$\boxed{\begin{array}{c} u = 2-y \\ du = u' \cdot dy \\ \parallel \\ -1 \end{array}}$

$\int \frac{1}{2-y}\,dy = \int \frac{1}{u} \cdot \frac{du}{-1}$

$= -\ln|u| + C$

$$\frac{1}{2-y}\, y' = 4t$$

$$\int \frac{1}{2-y}\,dy = \int 4t\,dt$$

$$\frac{1}{-1} \cdot \ln|2-y| = 2t^2 + C \quad | \cdot (-1)$$

$$\ln|2-y| = -2t^2 - C \quad | e^{\cdot}$$

$$|2-y| = e^{-2t^2 - C} = e^{-2t^2} \cdot e^{-C}$$

$$2-y = (\pm e^{-C})\, e^{-2t^2} = K e^{-2t^2}$$

$$\underline{\underline{y = 2 - K e^{-2t^2}}}$$

b) $\underbrace{y^2 - 2t}_{p} + \underbrace{2yt}_{q}\, y' = 0 \ , \quad y(1) = 2$

$\cancel{y' = \frac{2t - y^2}{2yt}}$

Exact? $\quad h'_t = y^2 - 2t \Rightarrow \underline{h = y^2 t - t^2 + C(y)}$

$h'_y = 2yt$

$\underbrace{2yt - \cancel{0} + C'(y)}_{h'_y} = 2yt$

$C'(y) = 0$, $C(y) = 0$

Yes

⇓

$h = y^2 t - t^2 = C \qquad \underline{y(1) = 2:} \qquad 2^2 \cdot 1 - 1^2 = C$

$t = 1, y = 2 \qquad \underline{C = 3}$

$y^2 t - t^2 = 3$

$\frac{y^2 t}{t} = \frac{t^2 + 3}{t}$

$y^2 = \frac{t^2+3}{t} \quad \longrightarrow \quad \underline{\underline{y = \sqrt{\frac{t^2+3}{t}}}}$

(c) $\underline{y}' = A \cdot \underline{y} + \underline{b} \qquad A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\underline{y} = \underline{y_h} + \underline{y_p} = \underline{\underline{C_1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}}}$$

$\underline{y_h}$: Eigenval. of A

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\underline{\lambda_1 = 2}, \quad \underline{\lambda_2 = -1}$

$\underline{E_2}$: $\begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \to \underline{v_1} = \underline{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}$

$\underline{E_{-1}}$: $\begin{pmatrix} 3 & 0 \\ \cancel{1} & \cancel{0} \end{pmatrix} \qquad \underline{v_2} = \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$$\Rightarrow \underline{y_h} = \underline{C_1 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot e^{-t}}$$

$\underline{y_p}$: $\underline{0} = A\underline{y} + \underline{b}$

$A\underline{y} = -\underline{b}$

$\left(\begin{array}{cc|c} 2 & 0 & 2 \\ 1 & -1 & -1 \end{array}\right) \rceil{-\tfrac{1}{2}}$

$\to \left(\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & -1 & -2 \end{array}\right) \qquad \begin{array}{ll} 2y_1 = 2 & y_1 = \underline{1} \\ -y_2 = -2 & y_2 = \underline{2} \end{array}$

$y_p = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

d) $y' + ty^2 = t$

$y' = t - ty^2 = t \cdot (1 - y^2) \qquad$ sep.

$\frac{1}{1-y^2} y' = t \qquad \frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y} \quad \Big| \; (1-y^2) \cdot$

$\uparrow$ $(1+y)(1-y)$

$\int \frac{1}{1-y^2} dy = \int t \, dt = \frac{1}{2} t^2 + C$

$1 = A \cdot (1+y) + B \cdot (1-y)$

$\int \frac{1/2}{1-y} dy + \int \frac{1/2}{1+y} dy = \frac{1}{2} t^2 + C \quad \Big| \cdot 2$

$y = 1: \quad 1 = A \cdot 2 + B \cdot 0$

$y = -1: \quad 1 = A \cdot 0 + B \cdot 2$

$A = 1/2$, $B = 1/2$

$\int \frac{1}{1-y} dy + \int \frac{1}{1+y} dy = t^2 + 2C$

$-\ln|1-y| + \ln|1+y| = t^2 + 2C \quad \Big| \; e^{\cdot}$

$$e^{-\ln|1-y| + \ln|1+y|} = e^{t^2+2c}$$

$$\frac{|1+y|}{|1-y|} = e^{t^2} \cdot e^{2c}$$

$$\frac{1+y}{1-y} = Ke^{t^2} \quad | \cdot (1-y)$$

$$1+y = Ke^{t^2} \cdot (1-y)$$

$$y(1+Ke^{t^2}) = Ke^{t^2} - 1$$

$$y = \frac{Ke^{t^2}-1}{1+Ke^{t^2}}$$

Other topics that are important but not covered by this final exam: (or not covered to a high degree)

I Linear algebra (matrices and vectors)

- diagonalization
- orthogonal diagonalization

II Optimization (max/min problems)

- unconstrained optimization, especially max/min $f(u)$ where $u = u(\underline{x})$ (composite functions)
- Kuhn-Tucker problems (see next page)
- envelope theorems

III Differential/difference equations:

- equilibrium states and stability
- difference equations incl. systems

Kuhn-Tucker problems:

std form:

max $f(\underline{x})$ when $g(\underline{x}) \leq a$

Kuhn-Tucker conditions:

FOC + C + CSC

$\mathcal{L}'(\underline{x}) = \underline{0}$ $\qquad \mathcal{L} = f(\underline{x}) - \lambda(g(\underline{x}) - a)$

$g(\underline{x}) \leq a$

$\lambda \geq 0$ and $\lambda = 0$ if $g(\underline{x}) < a$