	Plan	Textbook
1	Linear systems and their geometry	[E] 1.1 - 1.2
2	Gaussian elimination	[E] 1.3 - 1.5
3	Rank of a matrix	[E] 1.6
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① Linear systems and their geometry

Ex: $x+y+z+w=5$
$2x-3y+z-w=0$
$x+6y+2z+4w=17$

3×4 linear system in std form
↑ #eqns ↑ #var's

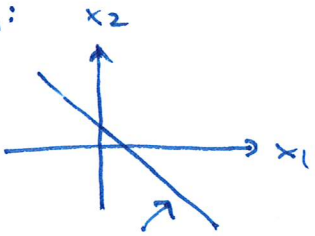
Defn: A linear eqn in n variables $x_1, x_2, \dots, x_n$ has the form

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \dots + a_n x_n = b$$

It is called degenerate if $a_1 = a_2 = \dots = a_n = 0$,
non-degenerate otherwise

Geometry:

$n=2$



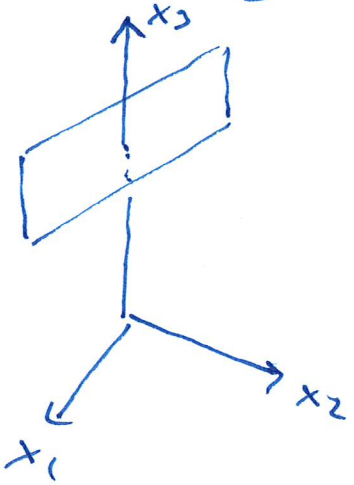
$a_1x_1 + a_2x_2 = b$
line

non-degenerate case

$x_2 = \dfrac{b - a_1x_1}{a_2}$

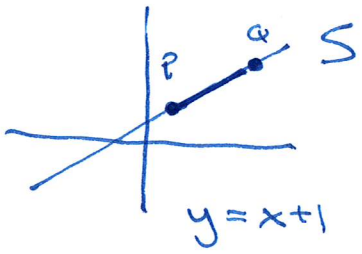
$0 \cdot x_1 + 0 \cdot x_2 = b$
degenerate case

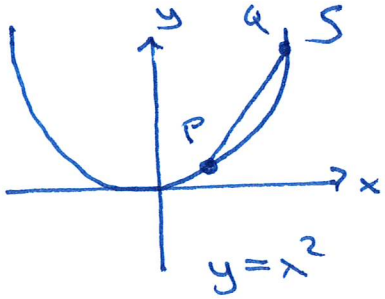
$n=3$: $a_1x_1 + a_2x_2 + a_3x_3 = b$ (non-degenerate case)



Def: A set S has the **linear property** if:

For all points P and Q in S with $P \neq Q$, the line segment $[P,Q]$ is contained in S





Fact: For any linear system, the set S of solutions has the linear property.

Thm: Any linear system has either

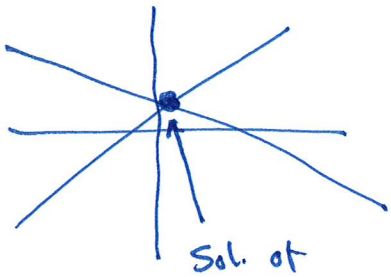
i) no solutions — inconsistent

ii) one unique solution

iii) infinitely many solutions

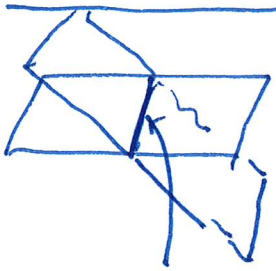
} consistent

2×2 lin. systems:



Sol. of syst

2×3 lin. syst:



Sol. of syst

3×3 lin. syst:

Sol of syst = a pt.

(2) Method: Gaussian elimination

Defn: An $m \times n$ linear system in std form is a system of eqn's of the form

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right.$$

m eqn's

n variables

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

coeff. matrix ($m \times n$)

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

augmented (extended) matrix

Def: Elementary row operations

- switch two rows
- multiply a row with $c \neq 0$
- add a multiple of one row to another row

An **echelon form**:

pivot: first non-zero entry in a row

echelon form:

- all zero rows are in the bottom of the matrix
- each pivot is further to the right than the pivots in the rows above

Ex:

$$\begin{aligned} x+y+z+w &= 5 \\ 2x-3y+z-w &= 0 \\ x+6y+2z+4w &= \cancel{17}\ 15 \end{aligned}$$

$$\left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & \cancel{17}\ 15 \end{array}\right) \cdot(-2) \leftarrow +$$

$$\rightarrow \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 1 & 6 & 2 & 4 & \cancel{17}\ 15 \end{array}\right) \cdot(-1) \leftarrow$$

$$\rightarrow \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 5 \\ 0 & \textcircled{-5} & -1 & -3 & -10 \\ 0 & 5 & 1 & 3 & \cancel{12}\ 10 \end{array}\right) \cdot 1 \leftarrow$$

$$\rightarrow \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 5 \\ 0 & \textcircled{-5} & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & \cancel{2}\ 0 \end{array}\right)$$

echelon form

$$\begin{aligned} x+y+z+w &= 5 \\ -5y-z-3w &= -10 \\ 0 &= 2 \end{aligned}$$

no solutions

New Ex:

$$\left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 5 \\ 0 & \textcircled{-5} & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

$$\begin{aligned} x+y+z+w &= 5 \\ -5y-z-3w &= -10 \end{aligned}$$

x, y: basic variables ← pivots

z, w: free variables ← no pivots

Back substitution:

$$-5y = z+3w-10 \Rightarrow y = -\frac{1}{5}z - \frac{3}{5}w + 2$$

$$x = -y-z-w+5$$

$$= -\left(-\frac{1}{5}z - \frac{3}{5}w + 2\right) - z - w + 5 = 3 - \frac{4}{5}z - \frac{2}{5}w$$

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Solutions:

$(x,y,z,w) = \left(3 - \frac{4}{5}z - \frac{3}{5}w,\ 2 - \frac{1}{5}z - \frac{3}{5}w,\ z,\ w\right)$ where z, w are free

$= \left(3 - \frac{4}{5}s - \frac{3}{5}t,\ 2 - \frac{1}{5}s - \frac{3}{5}t,\ s,\ t\right)$ (s, t are parameters)

Facts:

i) Any matrix can be transformed into an echelon form using elementary row operations.

ii) An echelon form is not unique, but the pivot positions are unique.

iii) The pivot positions determine the number of solutions:

- pivot in last column: no solutions
- no pivot in last column:
  - no free var's: one soln
  - at least one free var.: infinitely many solutions

Note: Elementary row operations preserve solutions

Summary: Gaussian elimination

1) Write down the augmented matrix
2) Transform it into an echelon form using elementary row op.'s
3) Use back substitution to solve the linear system.

③ Rank of a matrix

Def: The rank of a matrix is the number of pivot positions in an echelon form.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} -2 \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

$\operatorname{rk} A = 3$

$(A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) -2$

$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$ $\quad \operatorname{rk}(A|\underline{b}) = 3$

Fact: If $(A|\underline{b})$ is the augmented matrix of an $m \times n$ lin. system, then:

i) $\operatorname{rk}(A|\underline{b}) \neq \operatorname{rk} A \Rightarrow$ no solutions

ii) $\operatorname{rk}(A|\underline{b}) = \operatorname{rk} A \Rightarrow$ the system is consistent with $n - \operatorname{rk}(A)$ free var's

(4) Homogeneous linear systems

Def. An $m \times n$ linear system with augmented matrix $(A|\underline{b})$ is called **homogeneous** if $\underline{b} = \underline{0}$.

Ex:

$$\begin{aligned} x + y - z &= 0 \\ x + 2y + 4z &= 0 \\ 2x - 3y + 6z &= 0 \end{aligned}$$

homogeneous lin. system.

Note: $x = y = z = 0$ is a solution (the trivial sol'n)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & -3 & 6 \end{pmatrix} \begin{matrix} \\ -1 \\ -2 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -5 & 4 \end{pmatrix} \quad 5$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 19 \end{pmatrix} \Rightarrow \begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$$

is the unique solution

$\operatorname{rk} A = 3 \quad n = 3$

- no free var's: $n = \operatorname{rk} A$ — $(0,0,0)$ is the unique solution
- at least one free variable: $n > \operatorname{rk} A$ — $(0,0,0)$ is one of inf. many solutions

$$\begin{aligned} x + y + z &= 0 \quad & x &= 0 \\ y + 3z &= 0 \quad & y &= 0 \\ 19z &= 0 \quad & z &= 0 \end{aligned}$$

Ex: $x+y+2z=0$

$3x-y-z=0$

$5x+y+3z=0$

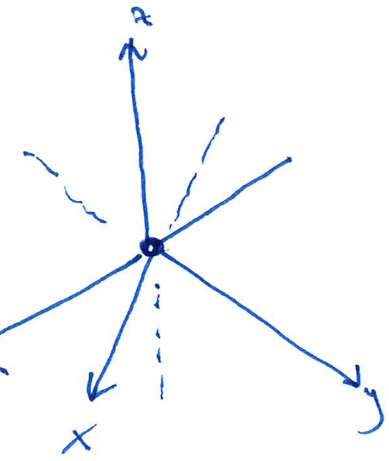
$$A=\begin{pmatrix}\textcircled{1} & 1 & 2\\ 3 & -1 & -1\\ 5 & 1 & 3\end{pmatrix}\begin{matrix}\\ -3\\ -5\end{matrix}$$

$$\to\begin{pmatrix}\textcircled{1} & 1 & 2\\ 0 & \textcircled{-4} & -7\\ 0 & -4 & -7\end{pmatrix}-1$$

$$\to\begin{pmatrix}\textcircled{1} & 1 & 2\\ 0 & \textcircled{-4} & -7\\ 0 & 0 & 0\end{pmatrix}\qquad \mathrm{rk}A=2$$

free var's = 3−2 = 1
(degrees of freedom)

(x,y basic, z free)



$x+y+2z=0$ $\quad x=-\left(-\frac{7}{4}z\right)-2z=-\frac{1}{4}z$

$-4y-7z=0$ $\quad -4y=7z$ $\quad y=-\frac{7}{4}z$

$(x,y,z)=\left(-\frac{1}{4}z,-\frac{7}{4}z,z\right)$ z free

$=\left(-\frac{1}{4}t,-\frac{7}{4}t,t\right)$

Geometric description of the solutions: S

A line in the 3-dim. space that goes through the origin

$n-\mathrm{rk}(A)=1$ (1-dim. sol'n-set)

$n=3$ (var's)

homogeneous (trivial sol'n)

dim = 0: • pt

dim = 1: / line

dim = 2: ▱ plane

dim > 2: hyperplane