	Plan	Textbook
1	Separable differential equations	[E] 7.1 - 7.4
2	Linear first order differential equations	[E] 7.5 - 7.6
3	Exact differential equations	[E] 7.7
4	Equilibrium states and stability	[E] 7.8
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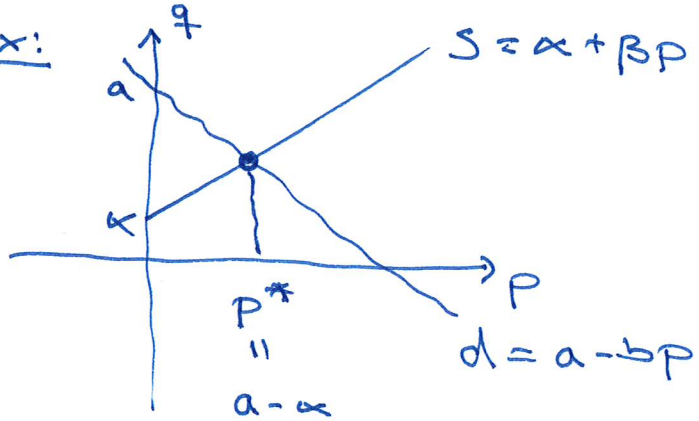
Plenary Session 3:
thu at 16-19
Lecture 7-9

Ex:



$p^* = \frac{a-\alpha}{b+\beta}$ eq. price

$p > p^*$: p decrease $p' < 0$
$p(t)$
$p < p^*$: p increase $p' > 0$

Model:

$p'(t) = k \cdot (d - s)$

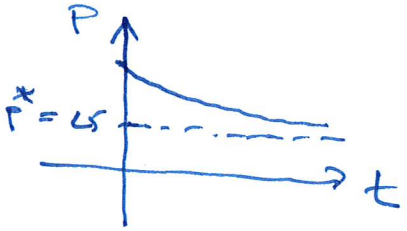
$\boxed{p' = k(d-s)}$

$p' = k[(a-bp) - (\alpha + \beta p)]$

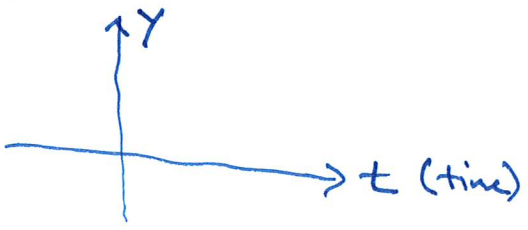
$= k(a-\alpha) - k(b+\beta)p$

Ex: $\boxed{p' = 50 - 2p}$ $p^* = 25$

$p'(t) = 50 - 2p(t)$



First order differential equations

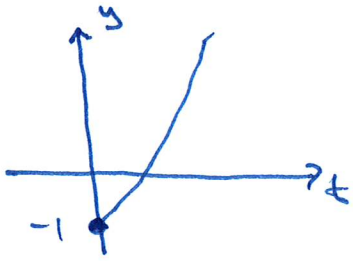


Def. A differential equ. is an equ. that involves an unknown function $y = y(t)$ and its derivatives. The order of the diff. equ. is the highest order derivative in the equ.

Ex: $y' = 4t+1$, $y(0) = -1$

$$y = \int 4t+1 \, dt = \underline{2t^2+t+C}$$

general solution



Ex: ~~$y' = 50-2y$~~

~~$y = \int 50-2y \, dt$~~

$y = 2t^2+t+C$

$-1 = 2\cdot 0^2+0+C$

$C = -1 \Rightarrow y = \underline{2t^2+t-1}$

particular solution

① Separable differential equations

A first order diff. eqn. is called **separable** if it can be written

$$y' = f(t)\cdot g(y)$$

Ex: $y' = 50-2y = \underbrace{2}_{f(t)}\cdot\underbrace{(25-y)}_{g(y)} \quad | : (25-y)$

$$\frac{1}{25-y}\, y' = 2 \quad | \int \dots dt$$

$$\int \frac{1}{25-y}\cdot y' \, dt = \int 2 \, dt$$

$y = y(t)$, $dy = y'\cdot dt$

$$\int \frac{1}{25-y} \, dy = 2t+C$$

$u = 25-y$, $du = -1\cdot dy$

$$\int \frac{1}{u}\cdot\frac{du}{-1} = -1\int \frac{1}{u}\,du = -\ln|u|+C$$

$$= -\ln|25-y|+C_1 = 2t+C_2$$

$$-\ln|25-y| = 2t+C_2-C_1 \quad |\cdot(-1)$$

$$\ln|25-y| = -2t+C_1-C_2 \quad |e^{\cdot}$$

$$|25-y| = e^{-2t+C_1-C_2}$$

$$25-y = \pm e^{-2t}\cdot e^{C_1-C_2}$$

$$25-y = K\cdot e^{-2t} \quad (K = \pm e^{C_1-C_2})$$

$$y = \underline{25-K\cdot e^{-2t}}$$

general solution, explicit form

General method

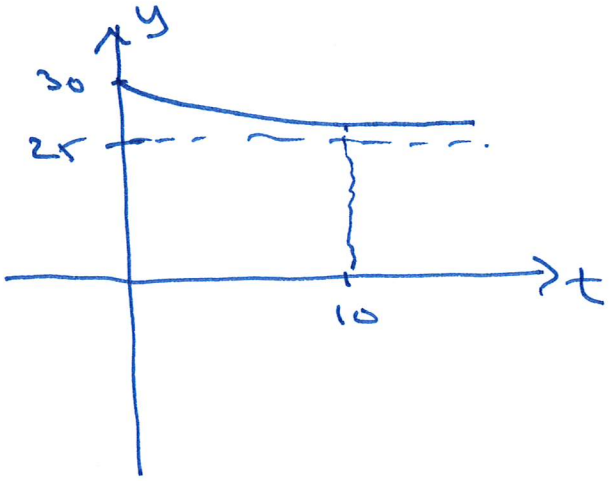
$y' = f(t)\cdot g(y)$

$\frac{1}{g(y)}\cdot y' = f(t)$

$\int \frac{1}{g(y)} y' \, dt = \int f(t)\,dt$

$\int \frac{1}{g(y)}\, dy = \int f(t)\,dt$

- solve both int.
- solve for y



$y' = 50 - 2y$, $y(0) = 30$

$y(t) = 25 - K \cdot e^{-2t}$

$30 = 25 - K \cdot e^{0} = 25 - K$

$K = -5$

Particular solution:

$y(t) = 25 + 5 \cdot e^{-2t}$

$y(10) = 25 + 5 \cdot e^{-20}$

Fact: Any first order differential equation has a general solution that depends on one undetermined coefficient.

Ex: $y' = \underbrace{0.15}_{f(t)}\underbrace{y}_{g(y)}$ is separable $\rightarrow \dots \rightarrow y = K \cdot e^{0.15t}$

$y' = y + t$ is not separable, but it is linear.

(2) Linear first order differential equations

Defn. A first ord. diff. eqn. is called linear if it can be written $\boxed{y' + a(t) \cdot y = b(t)} \Longleftrightarrow y' = \underbrace{b(t) - a(t)y}_{\text{linear in } y}$

Ex: $y' = y + t$ $\quad a(t) = -1$, $b(t) = t$

$y' - y = t \quad | \cdot u$

$u \cdot y' - u \cdot y = u \cdot t$

Want: $(uy)' = u \cdot t \quad | \int \dots dt$

$u = u(t)$:

integrating factor

$(uy)' = u' \cdot y + u \cdot y'$

$\Rightarrow u' = -u$

$$uy = \int u(t)\cdot t\,dt$$

$$y = \frac{1}{u(t)}\cdot\int u(t)\cdot t\,dt$$

$$y = \frac{1}{e^{-t}}\int e^{-t}\cdot t\,dt$$

$$= e^{t}\left(-te^{-t}-e^{-t}+C\right)$$

$$y = -t-1+Ce^{t}$$

Integrating factor:

$$u = e^{-t}$$

$$u' = e^{-t}\cdot(-1) = u\cdot(-1)$$

$\int t\cdot e^{-t}\,dt$:

Integration by parts

$$\int u'v\,dt = uv - \int uv'\,dt$$

$$\int \underbrace{t}_{v}\cdot\underbrace{e^{-t}}_{u'}\,dt = -e^{-t}\cdot t - \int -e^{-t}\cdot 1\,dt$$

$u = -e^{-t}$ $\quad v = t$

$u' = e^{-t}$ $\quad v' = 1$

$$= -te^{-t} + \int e^{-t}\,dt$$

$$= -te^{-t} - e^{-t} + C$$

Method: Integrating factor

$$y' + a(t)y = b(t) \quad |\cdot u$$

$$uy' + a(t)uy = b(t)\cdot u$$

$$(uy)' = b(t)\cdot u(t) \quad |\int \ldots dt$$

$$uy = \int b(t)\cdot u(t)\,dt$$

$$y = \frac{1}{u(t)}\cdot\int u(t)\cdot b(t)\,dt$$

$$(uy)' = u'y + uy' = uy' + a(t)uy$$

$$\Uparrow$$

$$u' = a(t)u$$

Integrating factor: $u = e^{\int a(t)\,dt}$

$$\Rightarrow u' = e^{\int a(t)\,dt}\cdot a(t) = u\cdot a(t)$$

Ex: $y' - y = t$

$a(t) = -1$

$\int a(t)\,dt = \int -1\,dt = -t + C$

$u = e^{-t+C} = e^{-t}$

Alternative method: Superposition $y = y_h + y_p$

$y' + ay = b(t)$ (with a = const.)

y_h: general solution of $y' + ay = 0$

y_p: particular solution of $y' + ay = b(t)$

Ex: $y' - y = t$

Superposition: $y = y_h + y_p = C \cdot e^t + t - 1 = \underbrace{C \cdot e^t}_{y_h} \underbrace{- t - 1}_{y_p}$

y_h: $y' - y = 0$

$y' = y$ $\quad y_h = C \cdot e^t$

y_p: $y' - y = t$ ← linear expr.

Try $y = At + B$, $y' = A$

$A - (At + B) = t$

$-At + A - B = t$

$A = -1$ $\quad A = B = -1$

$y_p = -t - 1$

(3) Exact differential equations

Motivation:

$h(t, y) = C \quad | \frac{d}{dt}$

$h'_t + h'_y \cdot y' = 0$

exact differential eqn.

Ex: $\underbrace{3t^2 - y}_{h'_t} \underbrace{- t}_{h'_y} y' = 0$

① $h'_t = 3t^2 - y$

② $h'_y = -t$

① $h'_t = 3t^2 - y$

$\Rightarrow h = \int 3t^2 - y \, dt$

$h = t^3 - yt + C(y)$

② $h'_y = (t^3 - yt + C(y))'_y = -t + C'(y) = -t$

$C'(y) = 0$

$C(y) = C$

Concl:

The differential eqn. is exact and the general solution is

$h(t, y) = C \Longrightarrow t^3 - yt + C_1 = C_2$

Ex: $3t^2 - y - ty' = 0$

$\frac{ty'}{t} = \frac{3t^2 - y}{t}$

$y' = \frac{3t^2 - y}{t} = 3t - \frac{1}{t}y$

- not separable

$t^3 - yt = K \quad (K = C_2 - C_1)$

$\frac{-yt}{-t} = \frac{K - t^3}{-t}$

$y = t^2 - \frac{K}{t}$ general solution in explicit form

- linear : $y' + \frac{1}{t} \cdot y = 3t \mid \cdot t$

$a(t) = 1/t$

$b(t) = 3t$

$ty' + y = 3t^2$

$(ty)' = 3t^2 \mid \int \cdot dt$

$\frac{ty}{t} = \int 3t^2 dt = \frac{t^3 + K}{t}$

Int. factor:

$\int a(t)dt = \int \frac{1}{t} dt$

$= \ln|t| + C$

$u = e^{\ln|t|} = |t|$

$= t$

$y = t^2 + \frac{K}{t}$

In general:

A differential eqn. is called exact if it can be written in the form $p(t,y) + q(t,y) \cdot y' = 0$ such that there is a fn. $h = h(t,y)$ with $h'_t = p$, $h'_y = q$. Then the general solution is $h(t,y) = C$.

(4) Equilibrium states and stability

Def. A first order diff. equ. is called **autonomous** if it can be written $y' = F(y)$.

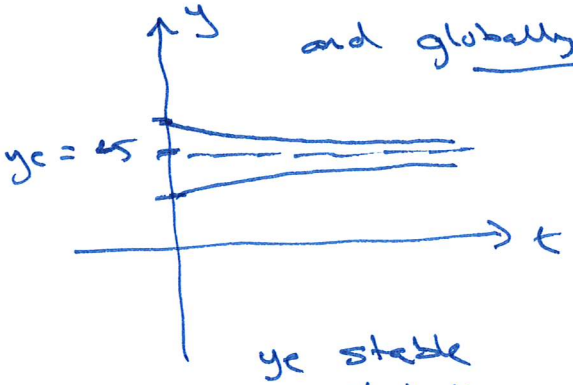
Ex: $y' = 50 - 2y$ is autonomous $y_e = 25$ eq. state

y_e is an **equilibrium state** if $F(y_e) = 0$

y_e is called **stable** if y_0 is close to y_e, then $y(t) \to y_e$ as $t \to \infty$

and **globally asymptotically stable** if $y(t) \to y_e$ for any y_0

unstable if it is not stable



y_e stable
globally as. stable

Stability Thm:

$F'(y_e) < 0 \Rightarrow y_e$ stable

$F'(y_e) > 0 \Rightarrow y_e$ unstable

$y' = 50 - 2y$

$y = 25 + C \cdot e^{-2t}$