	Plan	Textbook
1	Span of vectors and linear independence	[E] 2.1 - 2.3, 2.6
2	Vector spaces, bases and dimensions	[E] 2.7
3	Orthogonality and orthogonal projections	[E] 2.4 - 2.5,
	Problems from the textbook	[E] 2.1 - 2.16

Quick review: Key method: Gaussian elimination
Key notions: Pivot positions, free variables, rank, homogeneous linear systems, trivial solutions

① Span of vectors and linear independence

An n-vector $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ (column vector)

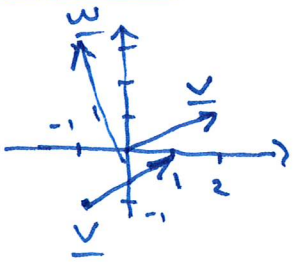
$\mathbb{R}^n$: space of all n-vectors
Euclidean n-space

Operations:
addition (subtraction) $\underline{v}+\underline{w}$, $\underline{v}-\underline{w}$
Scalar multiplication $r \cdot \underline{v}$
length of a vector $\|\underline{v}\| = \sqrt{v_1^2+v_2^2+\dots+v_n^2}$
inner product (dot product) $\underline{v}\cdot\underline{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n$

$\underline{v} \perp \underline{w}$ ($\underline{v}$ and $\underline{w}$ are orthogonal)
if and only if $\underline{v}\cdot\underline{w} = 0$

Geometric repr. of vectors:

Ex: $\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

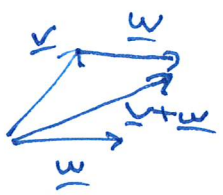


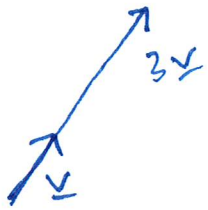
$\|\underline{v}\| = \sqrt{5}$

$\underline{v} \perp \underline{w}$ means that the angle between $\underline{v}$ and $\underline{w}$ is 90°

$\underline{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

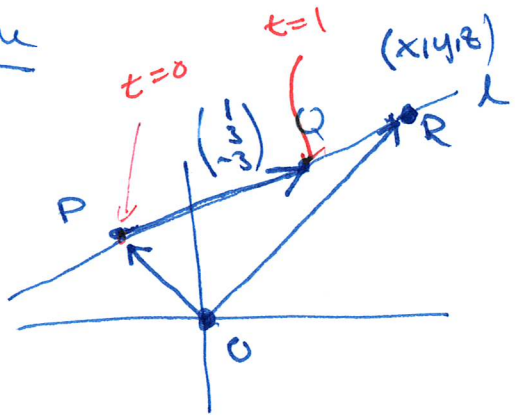
$\underline{v}\cdot\underline{w} = -2+3 = 1 \neq 0$
$\Rightarrow \underline{v}$ is not orth. to $\underline{w}$





Parametric representation of a line

Ex: l is the line through the points $(1,1,2)$, $(2,4,-1)$.
P Q



$$\vec{OR} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \vec{OR} = \vec{OP} + \vec{PR}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t \cdot \vec{PQ} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \qquad \begin{aligned} x &= 1 + t \\ y &= 1 + 3t \\ z &= 2 - 3t \end{aligned}$$

Span of vectors:

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ n-vectors

Defn: A linear combination of the vectors $\underline{v}_1, \ldots, \underline{v}_r$ is an expression of the form

$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \ldots + c_r \cdot \underline{v}_r$$

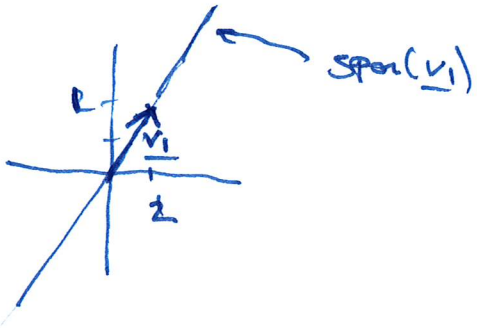
where $c_1, c_2, \ldots, c_r$ are given numbers (scalers)

The span of the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ is the set of all linear combinations of these vectors:

$$\text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r) = \{ c_1\underline{v}_1 + \ldots + c_r\underline{v}_r \text{ such that } c_1, c_2, \ldots, c_r \text{ scalers} \}$$

Ex: $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \text{span}(\underline{v}_1)$: $\quad c_1 \cdot \underline{v}_1 = c_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ all scalar multiples of $\underline{v}_1$

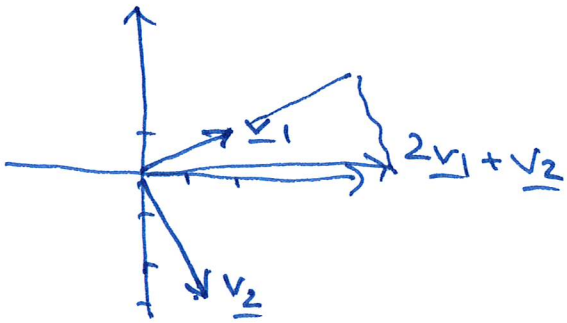
$r = 1$



Ex: $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\quad \text{span}(\underline{v}_1, \underline{v}_2)$: $\quad c_1\underline{v}_1 + c_2\underline{v}_2 = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

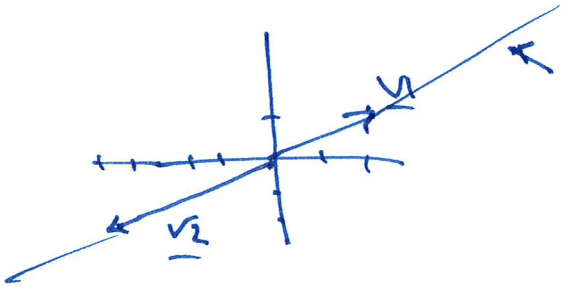
$$= \begin{pmatrix} 2c_1 + c_2 \\ c_1 - 3c_2 \end{pmatrix}$$

$r = 2$



$\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

$\underline{v}_2 = -2 \cdot \underline{v}_1$



$$c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 = c_1 \cdot \underline{v}_1 + (-2\underline{v}_1) \cdot c_2 = c_1\underline{v}_1 - 2c_2\underline{v}_1$$

$$= (c_1 - 2c_2) \cdot \underline{v}_1$$

Q: Is $\begin{pmatrix}7\\4\end{pmatrix}$ in the span of $\underline{v}_1=\begin{pmatrix}1\\2\end{pmatrix}$ and $\underline{v}_2=\begin{pmatrix}3\\-1\end{pmatrix}$?

$$c_1\cdot\begin{pmatrix}1\\2\end{pmatrix}+c_2\cdot\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}$$ vector equ.

$$\begin{pmatrix}c_1+3c_2\\2c_1-c_2\end{pmatrix}=\begin{pmatrix}7\\4\end{pmatrix}$$

$$\begin{aligned}c_1+3c_2&=7\\2c_1-c_2&=4\end{aligned}$$

$\underline{v}_1$ $\underline{v}_2$

$$\left(\begin{array}{cc|c}1&3&7\\2&-1&4\end{array}\right)\xrightarrow{-2}\left(\begin{array}{cc|c}1&3&7\\0&-7&-10\end{array}\right)$$

A: Yes, in exactly one way. Moreover, every 2-vector is in $\text{span}(\underline{v}_1,\underline{v}_2)$ in exactly one way

Concl: $\text{span}(\underline{v}_1,\underline{v}_2)=\mathbb{R}^2$

$$\begin{pmatrix}7\\4\end{pmatrix}=\tfrac{19}{7}\cdot\underline{v}_1+\tfrac{10}{7}\cdot\underline{v}_2$$

$$\begin{aligned}c_1+3c_2&=7\\-7c_2&=-10\end{aligned}$$

$$c_2=10/7$$

$$c_1=7-3\cdot(10/7)=19/7$$

Linear independence

$\underline{v}_1,\dots,\underline{v}_r$ n-vectors

Def. The vectors $\underline{v}_1,\underline{v}_2,\dots,\underline{v}_r$ are *linearly dependent* if one of vectors can be written as a linear combination of the others, and *linearly independent* otherwise.

Ex: Let $\underline{v}_1,\underline{v}_2,\underline{v}_3$ be vectors such that $\underline{v}_2=\underline{v}_1-\underline{v}_3$

Thm: i) $\underline{v}_1,\underline{v}_2,\underline{v}_3$ are lin. dependent

ii) $\text{span}(\underline{v}_1,\underline{v}_2,\underline{v}_3)=\text{span}(\underline{v}_1,\underline{v}_3)$

Fact: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r$ are lin. independent $\iff$ $x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + \ldots + x_r \cdot \underline{v}_r = \underline{0}$ has only the trivial solution $(x_1 = x_2 = \ldots = x_r = 0)$

$$\left(\underline{v}_1 \,|\, \underline{v}_2 \,|\, \ldots \,|\, \underline{v}_r \,\middle|\, \underline{0}\right)$$

Ex: $\underline{v}_2 = \underline{v}_1 - \underline{v}_3$

$\underline{v}_1 - \underline{v}_2 - \underline{v}_3 = \underline{0}$ has non-trivial soln $(1, -1, -1)$

Ex:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \underline{v}_4 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \rightsquigarrow x_1\underline{v}_1 + \ldots + x_4\underline{v}_4 = \underline{0}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 0 \\ 2 & -1 & 1 & 2 & 0 \\ 4 & 0 & -1 & 3 & 0 \end{array}\right) \begin{matrix} \\ \\ \end{matrix} \begin{matrix} \\ \\ \end{matrix} \begin{matrix} \\ \\ \end{matrix}

② Vectorspaces, bases and dimension

Vector space: Ex: $\mathbb{R}^n$

Defn: A subset V of $\mathbb{R}^n$ is a vector space if $V = \text{span}(\underline{v}_1, \ldots, \underline{v}_r)$ for some vectors $\underline{v}_1, \ldots, \underline{v}_r$ in $\mathbb{R}^n$

Column space: $\underline{v}_1, \ldots, \underline{v}_r \hookrightarrow A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_r)$

$V = \text{Col}(A) = \text{span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_r)$

column space

Defn: A base of a vector space V is a minimal spanning set of vectors in V, i.e. a linearly independent set of vectors that span V.

Ex: $A = \begin{pmatrix} \textcircled{1} & 3 & 1 & 5 \\ 2 & \textcircled{-1} & 1 & 2 \\ 4 & 0 & \textcircled{-1} & 3 \end{pmatrix}$ $\quad V = \text{Col}(A) = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$

$\underline{v}_1 \quad \underline{v}_2 \quad \underline{v}_3 \quad \underline{v}_4$ — vector space

Base: $\underline{v}_1, \underline{v}_2, \underline{v}_3$ $\quad \dim \text{Col}(A) = 3$

Defn: The dimension of a vector space V is the number of vectors in a base of V.

Fact: $\boxed{\dim \text{Col}(A) = \text{rk}\, A}$

Null(A): the set of solutions of $A\underline{x} = \underline{0} \Leftrightarrow (A|\underline{0})$

nullspace of A

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 & -1 \\ 1 & 3 & -1 & 3 \end{pmatrix}$ Find all solutions of the hom. lin. sys. with coeff-matrix A

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & -2 \\ 0 & 2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 + x_4 = 0$

$-2x_2 + 2x_3 - 2x_4 = 0$

x_3, x_4 free

$$\frac{-2x_2}{-2} = \frac{-2x_3 + 2x_4}{-2} \qquad x_2 = x_3 - x_4$$

$$x_1 = -x_2 - x_3 - x_4 = -(x_3 - x_4) - x_3 - x_4 \qquad x_1 = -2x_3$$

Solutions:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{w_1}$ $\underline{w_2}$

$\text{Null}(A) = \text{span}(\underline{w_1}, \underline{w_2})$ ← vector space

$\underline{w_1}, \underline{w_2}$ is a base of Null(A)

$\dim \text{Null}(A) = 2$

In general:

A m×n-matrix ⇝ $V = \text{Null}(A)$ is a vector space and $\dim \text{Null}(A) = n - rk(A)$

③ Orthogonality and orthogonal projection

$\underline{v}, \underline{w}$ n-vectors

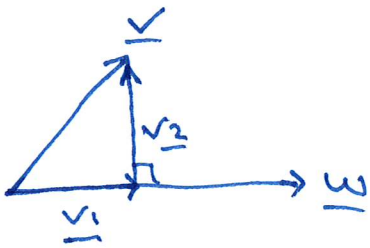
$\underline{v} \cdot \underline{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n$

Fact: $\underline{v} \perp \underline{w}$ (orthogonal) $\iff \underline{v} \cdot \underline{w} = 0$

Ex: $\underline{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 3 \cdot 1 + 1 \cdot (-2) + 1 \cdot (-1) = 0$

Ex: $\underline{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 1 \neq 0$



$\text{proj}_{\underline{w}}(\underline{v})$ = orthogonal proj. of $\underline{v}$ down on $\underline{w}$

Want: Decomposition $\underline{v} = \underline{v}_1 + \underline{v}_2$ where i) $\underline{v}_1$ is along $\underline{w}$ ii) $\underline{v}_2 \perp \underline{w}$

Def: $\text{proj}_{\underline{w}}(\underline{v}) = \underline{v}_1 = \dfrac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w}$

Method: $\underline{v}_1 = a \cdot \underline{w}$

$(\underline{v} - \underline{v}_1) \cdot \underline{w} = 0$

$(\underline{v} - a\underline{w}) \cdot \underline{w} = 0$

$\underline{v} \cdot \underline{w} - a \cdot \underline{w} \cdot \underline{w} = 0$

$\Rightarrow a = \dfrac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}}$

Ex: $\text{proj}_{\underline{w}}(\underline{v}) = \dfrac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \underline{w} = \dfrac{1}{4+16+1} \underline{w} = \dfrac{1}{21} \cdot \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$