	Plan	Textbook
1	Matrices and determinants	[E] 3.1 - 3.3
2	Minors and rank	[E] 3.4
	Problems from the textbook	[E] 3.1 - 3.15
	Exam problems	Final 11/2019 Q1, Midterm 01/2023 Q8, Midterm 01/2020 Q1,2,4,8 10/2022 Q7,8

Problem sessions / TA sessions : Fri 14-16 / Mon 16-18 in D3-020/030

Q. Review: key method:

Question about vectors ⇝ vector equ. ⇝ lin. sys.

Key notions:

span, linear independence, base / dimension, null space, column space

① Matrices

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ m (rows), n (columns)

man-matrix

Operations:

add./subtract. $A \pm B$

scalar mult. $c \cdot A$

matrix mult. $A \cdot B$

Ex: $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 13 \\ 1 & 4 \end{pmatrix}$

Note: $A \cdot B \neq B \cdot A$

Defn:

A is square if $m = n$

A is diagonal if it is square and $a_{ij} = 0$ if $i \neq j$

A is upper triangular if it is square and $a_{ij} = 0$ if $i > j$

$A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}$ diagonal

$A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}$ upper triangular

transpose: A^T

Ex: $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

Def. A is symmetric if $A^T = A$

inverse: A is called invertible if there is a matrix B such that $A \cdot B = I$, $B \cdot A = I$. In that case $B = A^{-1}$ is the inverse of A.

Def The identity matrix I is the diagonal matrix with 1's on the diagonal

Ex: $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Property: $A \cdot I = A$, $I \cdot A = A$

Fact:

A is invertible
⇕
A is square and $|A| \neq 0$

Determinants:

A ⇝ $\det(A) = |A|$

n×n-matrix — determinant of A, a number

n=2:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Methods for computing determinants: cofactor expansion / using Gaussian elim.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 6 \\ 1 & 3 & 6 \end{pmatrix}$

Cofactor expansion along the first row

$|A| = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$

$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$= +(12-16) - (6-12) + (4-6)$

$= -4 + 6 - 2 = 0$

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$+ - + \ldots$
$- + -$
$+ - +$
$\vdots$

M_{ij}: minor (i,j), determinant of the submatrix obtained by deleting row i, col j from A

Fact 1: Cofactor expansion of A along any row or column, gives the same result, $|A|$.

Fact 2: The determinant of an upper triangular matrix is the product of the diagonal entries.

Ex:

$$\begin{vmatrix} 2 & 7 & -14 \\ 0 & 3 & \sqrt{2} \\ 0 & 0 & -1 \end{vmatrix} = +2 \cdot \begin{vmatrix} 3 & \sqrt{2} \\ 0 & -1 \end{vmatrix} = 2\left(+3 \cdot (-1)\right)$$

$$= 2 \cdot 3 \cdot (-1) = \underline{\underline{-6}}$$

Fact 3: $|A^T| = |A|$

Using Gaussian elimination:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 3 & 4 & 6 \end{vmatrix} \begin{matrix} \\ \cdot(-1) \\ \cdot(-3) \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{vmatrix} \cdot(-1) \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} = |E| = \underline{\underline{0}}$$

E is upper triangular

$|E| = 1 \cdot 1 \cdot 0 = 0$

Fact 3:

All square echelon forms are upper triangular.

Fact 4: If $A \to B$ is an elementary row operation, then

$|B| = -|A|$ if you switch two rows

$|B| = c \cdot |A|$ if you multiply a row by $c \neq 0$

$|B| = |A|$ otherwise (add a multiple of one row to another row)

Ex:

$$\begin{vmatrix} 4 & 0 & 0 & -1 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 0 & 4 \end{vmatrix} \overset{\frac{1}{4}\left[\,\right]_4}{=} = +4 \cdot \begin{vmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & 3 & 2 \\ 0 & 2 & 3 \\ -1 & 0 & 0 \end{vmatrix}$$

$$\overset{\text{II}}{=} \quad = \ldots$$

$$\overset{-2/3\left[\,\right]_3}{=} \begin{vmatrix} 4 & 0 & 0 & -1 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 15/4 \end{vmatrix} = +4 \cdot \begin{vmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 15/4 \end{vmatrix} = \ldots$$

$$\overset{\text{II}}{=} \begin{vmatrix} 4 & 0 & 0 & -1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 5/3 & 0 \\ 0 & 0 & 0 & 15/4 \end{vmatrix} = 4 \cdot 3 \cdot \frac{5}{3} \cdot \frac{15}{4} = \underline{75}$$

Fact: $|A| = 0 \iff$ there are free variables in the homogeneous lin. sys. $A\underline{x} = \underline{0}$. $\iff$ the column vectors of A are linearly dependant

$\iff$ A is not invertible

A invertible (A^{-1} exists) $\iff |A| \neq 0 \iff$ no free variables in $A\underline{x} = \underline{0}$ $\iff$ the column vectors of A are linearly independent

Inverses:

A $n \times n$ matrix. Assume $|A| \neq 0$. Thm: $A^{-1} = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T}_{\text{adj}(A)}$

Ex:

$$\begin{aligned} x+y+z &= 4 \\ x+2y+4z &= 7 \\ 3x+4y+5z &= 11 \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix}$$

Matrix form: $A \cdot \underline{x} = \underline{b}$

$$|A| = 1 \cdot (-6) - 1(-7) + 1(-2)$$
$$= -6 + 7 - 2 = -1 \neq 0$$

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} -6 & 7 & -2 \\ -1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix}^T = - \begin{pmatrix} -6 & -1 & 2 \\ 7 & 2 & -3 \\ -2 & -1 & 1 \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b} \quad | \cdot A^{-1}$$
$$A^{-1} A \underline{x} = A^{-1} \underline{b}$$
$$I \underline{x} = A^{-1} \underline{b}$$
$$\underline{x} = A^{-1} \underline{b}$$

$$A^{-1} = \begin{pmatrix} 6 & 1 & -2 \\ -7 & -2 & 3 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 & 1 & -2 \\ -7 & -2 & 3 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} = \begin{pmatrix} 9 \\ -9 \\ 4 \end{pmatrix}$$

$x = 9$
$y = -9$
$z = 4$

(3) Minors and rank

A man-matrix

Defn. An **r-minor of A** is the determinant of an $r \times r$ - submatrix of A.

Ex: $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 3 \end{pmatrix}$

2×3-matrix

2-minors = maximal minors

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$ " $M_{12,12}$ $\quad \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$ " $M_{12,13}$ $\quad \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} = -2$ " $M_{12,23}$

1-minors

1 2 2

2 4 3

Ex: $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

2×2-matrix

2-minors = maximal minors = $|A| = 1$

1-minors: 1 1

1 2

Result: the rank of an $m \times n$ matrix is the maximal r such that an r-minor of A is non-zero.

Ex: $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad \mathrm{rk}\, A = 2$

2-minors: $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \quad \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

1-minors: $1, 2, 1, 2, 2, 3$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad 2\text{-minors} = |A| = 1 \neq 0$

$\mathrm{rk}\, A = 2$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & q \end{pmatrix}$ $\quad \mathrm{rk}\, A = \begin{cases} 2, & (p,q) \neq (1,1) \\ 1, & (p,q) = (1,1) \end{cases}$

2-minors:
$M_{12,12} = p - 1$
$M_{12,13} = q - 1$
$M_{12,23} = q - p$

$\mathrm{rk}\, A < 2$ if
$p - 1 = 0$
$q - 1 = 0$
$q - p = 0$
$\Rightarrow p = 1, q = 1$

1-minors: $1, 1, 1, 1, p, q$

Reformulation:

$\mathrm{rk}\, A < r$

$\Updownarrow$

all r-minors are zero

More on determinants

Fact 1: $|A \cdot B| = |A| \cdot |B|$

Fact 2: $|A| = 0$ in the following cases:

i) A has a zero row (or column)

ii) A has two equal rows (or columns)

iii) A has a row that is a lin. comb of the other rows (or a column that is a linear comb. of the other col's)

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 5 \end{vmatrix} = 0 \qquad R_3 = R_1 + R_2$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{vmatrix} = 0 \qquad C_3 = 2C_1 + C_2$$

Summary: A $n \times n$-matrix

$\lvert A \rvert \neq 0$	$\lvert A \rvert = 0$
A invertible	A not invertible
$A\underline{x} = \underline{b}$ has one solution for any $\underline{b}$; $\underline{x} = A^{-1}\underline{b}$	$A\underline{x} = \underline{b}$ has either no solutions or inf. many solutions for any $\underline{b}$
$\text{rk } A = n$	$\text{rk } A < n$
the col. vectors of A are linearly independent	the col. vectors of A are linearly dependent