	Plan	Textbook
1	Eigenvalues and eigenvectors	[E] 4.1 - 4.2
2	Diagonalization	[E] 4.3
	Textbook problems	[E] 4.1 - 4.7
	Exam problems	Final 11/2023 Q1c, 2, 01/2025 Q1d, 2ab, Midterm 10/2022 Q3,6, 01/2023 Q1-3,6

Mon at 16-19: Plenary Session 1: Problems from Lecture 1-4

Q. Review: Key method computing determinants and minors

Result:

A $n \times n$-matrix $\iff |A| \neq 0 \iff \operatorname{rk} A = n \iff$ the (cols) rows of A are linindependent $\iff A\underline{x} = \underline{0}$ has only the trivial solution

① Eigenvalues and eigenvectors

Ex: Unemployment

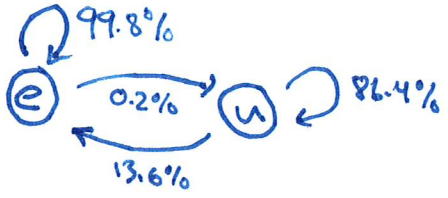
$\underline{v} = \begin{pmatrix} e \\ u \end{pmatrix}$ State vector

Ex: $\underline{v} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$ 10% unemployment

$0 \leq e, u \leq 1$

$e + u = 1$

$A = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}$



$\underline{v}_{t+1} = A \cdot \underline{v}_t$

$\begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} \begin{pmatrix} e_t \\ u_t \end{pmatrix}$

$e_{t+1} = 0.998\, e_t + 0.136\, u_t$

$u_{t+1} = 0.002\, e_t + 0.864\, u_t$

- all entries in A are between 0 and 1
- all column sums are 1

A $n \times n$-matrix $\quad A\underline{v} = \lambda \underline{v}$ (*)

$\underline{v}$: n-vector

λ: scalar (number)

(*) $A\underline{v} = \lambda \underline{v}$

$A\underline{v} - \lambda I \underline{v} = \underline{0}$

$(A - \lambda I)\underline{v} = \underline{0}$ (**)

Homogeneous linear system with parameter λ

Def: An eigenvalue of A is a number λ s.t. (*) has non-trivial solutions $\underline{v} \neq \underline{0}$.

$|A - \lambda I| = 0 \iff \lambda$ is an eigenvalue of A

(characteristic equation)

An eigenvector of A with eigenvalue λ is a non-zero solution $\underline{v}$ of (*).

E_λ = eigenspace of A with eigenvalue λ

= all solutions of (*)

= Null$(A - \lambda I)$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \lambda$

$A\underline{v} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \lambda \cdot \underline{v} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

$2x + y = \lambda x \quad 2x - \lambda x + y = 0$

$x + 2y = \lambda y \quad x + 2y - \lambda y = 0$

$(A - \lambda I)\underline{v} = \underline{0} \qquad \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$|A - \lambda I| \neq 0$: $\underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$|A - \lambda I| = 0$: int. many solutions for $\underline{v}$

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1^2 = 0$

$a_{11} + a_{22} + \dots + a_{nn} = \mathrm{tr}(A)$ $\qquad |A|$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0$

Eigenvalues of A $\rightsquigarrow$ $\lambda = 1$, $\lambda = 3$

Note:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad 2 \times 2$ $\quad \rightsquigarrow \quad |A - \lambda I| = \lambda^2 - \mathrm{tr}(A)\lambda + |A|$

$\lambda = 1$: $\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

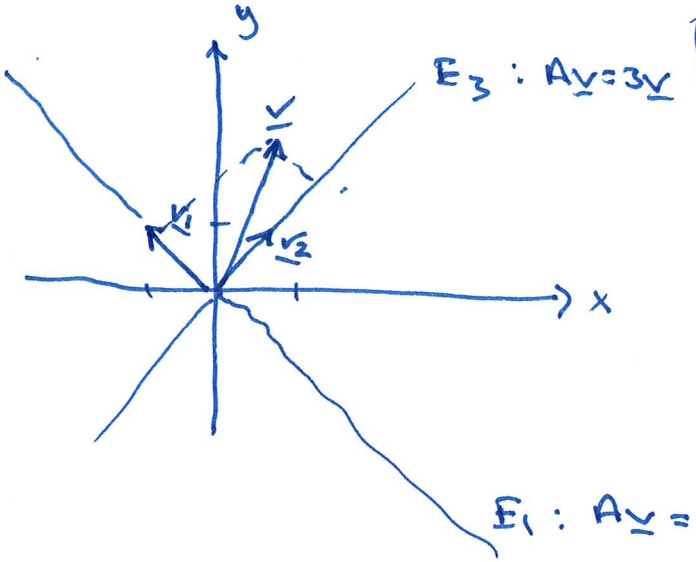
$\begin{pmatrix} \textcircled{1} & 1 \\ 0 & 0 \end{pmatrix}$ $\quad x + y = 0 \quad x = -y$

y free $\quad y = y$

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \underline{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvector for $\lambda = 1$: span$(\underline{u})$

$E_1 = \mathrm{span}(\underline{u})$ $\quad$ ($\underline{u}$ is a base of E_1)



$\lambda=3$: $\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad -x+y=0$, y free

$\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a base of E_3

$\underline{v} = \underline{v_1} + 2\underline{v_2}$

$A\underline{v} = A(\underline{v_1} + 2\underline{v_2}) = \underline{v_1} + 2 \cdot 3\underline{v_2}$

$= \underline{v_1} + 6\underline{v_2}$

General method for finding eigenvalues and eigenvectors of A:

A $n \times n$-matrix

i) Eigenvalues of A = solutions of $|A - \lambda I| = 0$ (Char. eqn.) (polynomial eqn. of degree n)

ii) Eigenvectors of A = solutions of $(A - \lambda I)\underline{v} = \underline{0}$ for each eigenvalue λ $\quad E_\lambda = \text{Null}(A - \lambda I)$

$\dim E_\lambda = \#$ free variables ≥ 1

Ex: i) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \lambda^2 - 0 \cdot \lambda + 1 = 0$

$\lambda^2 + 1 = 0$ $\quad \lambda^2 = -1$ $\quad$ no eigenvalues

ii) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \lambda^2 = 0$ $\quad \lambda = 0$ eigenvalue of mult. 2

$\lambda_1 = \lambda_2 = 0$

$\lambda \cdot \lambda = 0$

Ex:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \qquad \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$+(2-\lambda)\cdot\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\cdot(\lambda^2 - 2\lambda + 8) = 0$$

$\lambda = 2$ or $(\lambda+2)(\lambda-4) = 0$

$\lambda = -2$, $\lambda = 4$

Facts: A $n \times n$-matrix

i) There are at most n eigenvalues (counted with mult.)

ii) If λ is an eigenvalue of A with multiplicity m, then

$$1 \leq \dim E_\lambda \leq m$$

$\dim E_\lambda$ = # free var.

iii) If A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then

$$\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = |A|$$

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \mathrm{tr}(A)$$

Ex:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Notice: $\lambda = 1$ is an eigenvalue of multiplicity $m \geq 2$

$$A - 1I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\cdot(-1)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

two free var.

$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = ?$

$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A)$

$1 + 1 + \lambda_3 = 6 \quad \lambda_3 = 4$

Conclusion:

$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 4$

Alternative computation of eigenvalues of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} =$$

$$(2-\lambda)\cdot\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (2-\lambda)\left((2-\lambda)^2 - 1^2\right) - (2-\lambda-1) + (1-(2-\lambda))$$

$$= (2-\lambda)(\lambda^2-4\lambda+3) + (\lambda-1) + (\lambda-1)$$

$$= (2-\lambda)(\lambda-3)(\lambda-1) + 2(\lambda-1)$$

$$= (\lambda-1)\cdot\left[(2-\lambda)(\lambda-3)+2\right]$$

$$= (\lambda-1)\cdot(-\lambda^2+2\lambda+3\lambda-6+2)$$

$$= (\lambda-1)(-\lambda^2+5\lambda-4)$$

$$= -(\lambda-1)(\lambda^2-5\lambda+4)$$

$$= -(\lambda-1)(\lambda-1)(\lambda-4)$$

$$= -(\lambda-1)^2(\lambda-4) = 0$$

$\lambda=1$ multiplicity $m=2$

$\lambda=4$ — " — $m=1$

$\Rightarrow \lambda_1=\lambda_2=1,\ \lambda_3=4$

(2) Diagonalization

Defn An $n \times n$-matrix A is *diagonalizable* if there is an invertible matrix P such that $P^{-1}AP = D$ is diagonal.

Note: $A \cdot (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) = (A\underline{v}_1 | A\underline{v}_2 | \dots | A\underline{v}_n)$

$\underline{v}_1, \dots, \underline{v}_n$ eigenvectors of A

P

$= (\lambda_1 \underline{v}_1 | \lambda_2 \underline{v}_2 | \dots | \lambda_n \underline{v}_n)$

$= (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{pmatrix}$

$AP = PD \mid P^{-1}$ $\quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$

$P^{-1}AP = D$

Result: A is $n \times n$ matrix diagonalizable $\iff$ A has n linearly independent eigenvectors

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \lambda_1 = 1$ $\quad \underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ Base of E_1

$\lambda_2 = 3$ $\quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ — " — E_3

$\underline{v}_1$ and $\underline{v}_2$ are lin. independent

$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ $\Big\}$ $P^{-1}AP = D$, D is diag.

$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0$ (ok)

A is diag since $\dim E_1 + \dim E_3 = 2 = n$

Fact: Eigenvectors from different eigenspaces are always lin. independent

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\lambda_1 = \lambda_2 = 0$ $\lambda = 0$ $(m=2)$

E_0: $\begin{pmatrix} 0 & \textcircled{1} \\ 0 & 0 \end{pmatrix}$ $y = 0$, x free

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} 1 & ? \\ 0 & ? \end{pmatrix}$

$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\dim E_0 = 1 < n = 2$

A not diagonalizable

Fact: A $n \times n$-matrix

A diagonalizable $\iff$ i) A has n eigenvalues, counted with mult.

and

ii) For each eigenvalue λ, # free var. $= \dim E_\lambda = m$ (mult. of λ)

Note: If $m = 1$, then $\dim E_\lambda = m$ is automatic.

This means that if A has n distinct eigenvalues, it is diagonalizable.

Ex:

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\left.\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 4 \end{array}\right\}$ $\lambda = 1$ $(m=2)$

$\lambda = 1$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x + y + z = 0$, y, z free

$x = -y - z$, $y = y$, $z = z$

$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Base of E_1: $\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$P = \left(\underline{v}_1 \middle| \underline{v}_2 \middle| \underline{v}_3 \right) = \begin{pmatrix} -1 & -1 & \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix}$

A is diagonalizable

Fact: A symmetric $\Rightarrow$ A is diagonalizable

Ex: Computing powers of matrices

A $n \times n$-matrix. Want to compute A^{100} or A^N

Assume that A is diagonalizable with

P.I $P^{-1}AP = D$. Then:

$\cdot P^{-1}$ I $AP = PD$

$A = PDP^{-1} \Rightarrow A^N = (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$ (N times)

$$\boxed{A^N = PD^NP^{-1}}$$

Ex: $A = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$ $\lambda_1 = 1$, $\underline{v}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ base of E_1

$\lambda_2 = 0.2$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ -"- $E_{0.2}$

$$A^N = \underbrace{\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}}_{P} \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}^N \cdot \underbrace{\frac{1}{8}\begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}}_{P^{-1}} = \frac{1}{8}\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1^N & 0 \\ 0 & 0.2^N \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}$$

($1^N = 1$, $0.2^N \to 0$)

$$\to \frac{1}{8}\begin{pmatrix} 5 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 5 & 5 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 5/8 & 5/8 \\ 3/8 & 3/8 \end{pmatrix}$$

$N \to \infty$