	Plan	Textbook
1	Definiteness of quadratic forms	[E] 4.6 (Principal minors p. 110 - 116)
2	Unconstrained optimization	[E] 5.1 - 5.4
3	Convex and concave functions	[E] 5.5 - 5.6
	Textbook problems	4.12 - 4.14, 5.1 - 5.12
	Exam problems	Midterm 01/2023 Q4,5,7
		Mock midterm 10/2022 Q1-21

Remember: Midterm evaluation! Deadline Oct 3rd.

Review: Markov chains. Orthogonal diagonalization.

① Definiteness of quadratic forms

$$f(\underline{x}) = \underline{x}^T A \underline{x} = \underline{u}^T D \underline{u} = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \dots + \lambda_n u_n^2$$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ A symm. $n \times n$-matrix $P^T A P = D$ orth. diag. $\underline{x} = P \cdot \underline{u}$ $\underline{u} = P^T \cdot \underline{x}$

$\lambda_1, \dots, \lambda_n \geq 0 \leftrightarrow f$ pos. semidefin.
$\lambda_1, \dots, \lambda_n \leq 0$ f neg. semidefin.
otherwise f indefin.
$\lambda_1, \dots, \lambda_n > 0$ f pos. defin.
$\lambda_1, \dots, \lambda_n < 0$ f neg. defin.

$2u^2 + 4v^2 - 4w^2$ = f. pos. defin.

Ex: $f(x,y,z) = 3x^2 - 2xy + 3y^2 - 4z^2$

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 4 \\ \lambda_3 = 4 \end{matrix}$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Alternative method:

$u = \frac{1}{\sqrt{2}}(x+y)$
$v = \frac{1}{\sqrt{2}}(-x+y)$
$w = z$

A $n \times n$ symm. matrix

Δ_r **Def** An r-minor of A is principal when we keep the same rows as columns.

D_r It is called a leading principal r-minor if we keep the first r rows and cols.

Ex:

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \begin{matrix} D_1 = 3 \\ D_2 = 8 \\ D_3 = 32 \end{matrix}$$

(2) Unconstrained optimization

$$\max/\min f(x_1, x_2 \dots x_n) = f(\underline{x})$$

Defn. A **stationary** pt of f is a pt $\underline{x}$ such that $\boxed{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}$

FOC = first order condition

Fact: If $\underline{x}$ is a max/min pt for f, then either

i) $\underline{x}$ is a stationary pt

ii) $f'_{x_i}(\underline{x})$ is not defined for some i

iii) $\underline{x}$ is a boundary pt for D_f

Recall: D_f = domain of definition for f

Ex: $f(x,y,z) = x^3 - xyz + z^3$ $\quad D_f = \mathbb{R}^3$

$f'_x = 3x^2 - yz$ $\quad f'_y = -xz$ $\quad f'_z = -xy + 3z^2$

Method: $\max/\min f(\underline{x})$

i) Compute stat. pts of f
ii) Check if there are other possible max/min satisfying i) or iii)

} list of candidate pts

iii) Classify the candidate pts as local max / local min / saddle pts.

iv) Determine if local max / local min are max/min.

Result 1:

$D_1, D_2, \dots, D_n > 0 \iff A$ pos. defn.

$D_1 < 0, D_2 > 0, \dots \iff A$ neg. defn.

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $D_1 = 2$ +
$D_2 = 2 \cdot 4$ +
$D_3 = 2 \cdot 4 \cdot 4$ +

$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$ $D_1 = -2$ −
$D_2 = (-2)\cdot(-4)$ +
$D_3 = (-2)(-4)(-4)$ −

Ex:

$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ $D_1 = 1$ $\Delta_1 = 1, 1, 3$
$D_2 = 0$ $\Delta_2 = 0, -1, -1$
$D_3 = 0$ $\Delta_3 = 0$

" Minor

indefn

$\operatorname{rk} A = 2$: cannot use RRC.

Result 2:

$\Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \iff A$ pos. semidefn.

$\Delta_1 \leq 0, \Delta_2 \geq 0, \dots \iff A$ neg. semidefn.

otherwise $\iff A$ indefinite

Reduced rank criterion (RRC)

A n×n symm. $\operatorname{rk} A < n \iff |A| = 0$ " D_n

If $\operatorname{rk} A = r < n$, then:

i) If $D_1, D_2, \dots, D_r > 0$ then A pos. semidefn.

ii) If $D_1 < 0, D_2 > 0, \dots, (-1)^r \cdot D_r > 0$, then A neg. semidefn.

Ex:

$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$ $D_1 = 2$
$D_2 = 5$
$D_3 = 0$

$\operatorname{rk} A = 2$: $r = 2 < 3 = n \underset{\text{RRC}}{\Longrightarrow} A$ is pos. semidefn.

Ex: $f(x,y,z) = 7 - 3x^2 + 2xy - 3y^2 - 4z^2$ $\quad D_f = \mathbb{R}^3$

$f'_x = -6x + 2y = 0 \quad y = 3x \quad y = 0$

$f'_y = 2x - 6y = 0 \quad 2x - 6(3x) = 0 \quad -16x = 0 \quad x = 0$

$f'_z = -8z = 0 \quad z = 0$

Stat. pts: $(x,y,z) = (0,0,0)$ ← Cand pts

Def: A max pt for f is a pt $\underline{x}^*$ such that $f(\underline{x}^*) \geq f(\underline{x})$ for all pts $\underline{x}$, and min pt if $f(\underline{x}^*) \leq f(\underline{x})$ for all pts $\underline{x}$.

A local max pt for f is a pt $\underline{x}^*$ such that $f(\underline{x}^*) \geq f(\underline{x})$ for all pts $\underline{x}$ close to $\underline{x}^*$, and a local min if $f(\underline{x}^*) \leq f(\underline{x})$

— " —

A candidate pt that is neither local max nor local min is called a saddle pt.

Second derivative test:

If $\underline{x}^*$ is a stationary pt of f, then we look at

$$H = H(f)(\underline{x}^*) = \begin{pmatrix} f''_{x_1x_1}(\underline{x}^*) & f''_{x_1x_2}(\underline{x}^*) & \cdots \\ f''_{x_2x_1}(\underline{x}^*) & f''_{x_2x_2}(\underline{x}^*) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

i) If H is positive definite, then $\underline{x}^*$ is a local min.
ii) If H is negative definite, then $\underline{x}^*$ is a local max.
iii) If H is indefin. then $\underline{x}^*$ is a saddle pt.

$f'_x = -6x + 2y$

$f'_y = 2x - 6y$

$f'_z = -8z$

$$H(f) = \begin{pmatrix} -6 & 2 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & -8 \end{pmatrix} = Hf(0,0,0) = H$$

$\simeq 2A$

$D_1 = -6 < 0$

$D_2 = 32 > 0$

$D_3 = -8 \cdot 32 = -256 < 0$

$\Rightarrow$ H neg. def. $\Rightarrow$ $(0,0,0)$ local max

$f(x,y,z) = 7 - 3x^2 + 2xy - 3y^2 - 4z^2$

$f(0,0,0) = 7$

$-3x^2 + 2xy - 3y^2 - 4z^2 \leq 0$

quadr. form

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$ neg. def.

$D_1 = -3$

$D_2 = 8$

$D_3 = -32$

Concl: $(0,0,0)$ is max pt

$f(0,0,0) = 7$ is max value

In general:

$f(\underline{x})$ polynomial of degree ≤ 2

$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C \Rightarrow H(f) = 2A$

Ex: $f(x,y,z) = x^3 + y^3 + z^3 - 3x - 3y - 3z + 5$

$f'_x = 3x^2 - 3 = 0 \quad x^2 = 1 \quad x = \pm 1$

$f'_y = 3y^2 - 3 = 0 \quad y^2 = 1 \quad y = \pm 1$

$f'_z = 3z^2 - 3 = 0 \quad z^2 = 1 \quad z = \pm 1$

Stat. pts: $(\pm 1, \pm 1, \pm 1)$ = cand. pts.

$$H(f) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

$$Hf(1,1,1) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

pos. def: $(1,1,1)$ local min

$$Hf(-1,-1,-1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

neg. def. $(-1,-1,-1)$ local max

remaining six stat. pts: saddle pts

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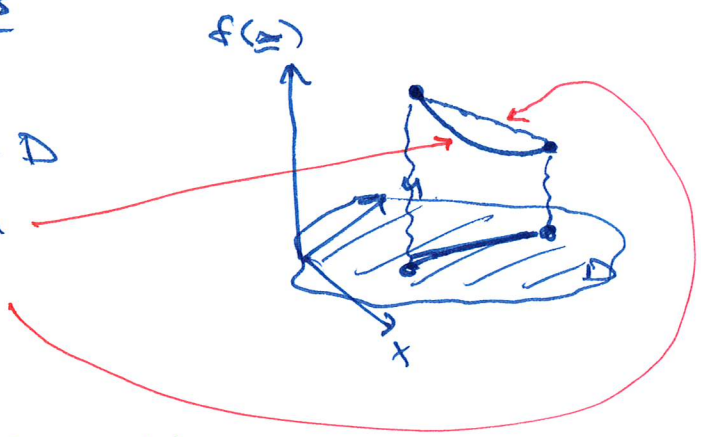
$f(1,1,1) = -1$ min value? $f(-1,-1,-1) = 4$ max value?

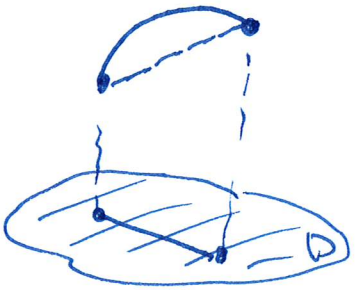
(3) Convex/concave functions



Defn: f defined on a convex set D is called **convex** if the graph is below the straight line (or on)

Fact: f is convex on D $\iff$ $Hf(\underline{x})$ is positive semidefinite for all $\underline{x}$ in D

Result: If f is convex, then any stationary pt of f is a min pt (global min).



Defn f defined on a convex set D is concave if the graph is above (or on) the line.

Fact: f concave $\iff$ $Hf(\underline{x})$ is negative semidefinite for all $\underline{x}$

Result: If f is concave, then any stationary pt of f is a max pt (global max).

Ex: $f(x,y,z) = x^3+y^3+z^3-3x-3y-3z+5$

$f'_x = 3x^2-3$
$f'_y = 3y^2-3$
$f'_z = 3z^2-3$

$H(f) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$ $\lambda_1 = 6x$, $\lambda_2 = 6y$, $\lambda_3 = 6z$

f is not convex, not concave

$f(1,1,1) = -1$ local min $\quad f(-10,0,0) = -1000+30+5 < -1$

$\Rightarrow$ No min

$f(-1,-1,-1) = 11$ local max $\quad f(10,0,0) = 1000-30+5 > 11$

$\Rightarrow$ No max

Range of f = the set of all possible values of f.

Ex: $f(x,y,z) = 7-3x^2+2xy-3y^2-4z^2$ $\quad f(0,0,0)=7$ max, no min

$V_f = (-\infty, 7]$

$f(x,y,z) = x^3+y^3+z^3-3x-3y-3z+5$ $\quad$ no min, no max

$V_f = (-\infty, \infty)$

Special case: $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

A pos semidefin: f convex $\quad H(f) = 2A \Rightarrow f$ has global min

A neg $-''-$: f concave $\quad -''- \Rightarrow f$ has global max

A indefin: $\quad H(f) = 2A$ indefin. $\Rightarrow$ All stat pts are saddle pts. $V_f = (-\infty, \infty)$