	Plan	Textbook
1	Constrained optimization	[E] 6.1 - 6.4
2	Second order conditions	[E] 6.4
	Textbook problems	6.1 - 6.2, 6.3ab, 6.4, 6.11
	Exam problems	Final 11/2019 Q4ab, 11/2022 Q2, 11/2023 Q1d, Q3

Plenary Session 2: Monday at 16-19 in A1-040
Problems from Lecture 5-7

Q. Review: $\max/\min f(\underline{x}) = f(x_1, x_2, \dots, x_n)$

(a) Candidate pts: $f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0$ (stationary pts)
FOC

(b) Second derivative test:

$\underline{x}^*$ stat. pt: $H(f)(\underline{x}^*)$ positive defn. $\Rightarrow \underline{x}^*$ local min
$H(f)$ is symmetric " negative " $\Rightarrow \underline{x}^*$ local max
indefn. $\Rightarrow \underline{x}^*$ saddle pt.

(c) Convex optimization:

f convex $\Leftrightarrow$ $H(f)(\underline{x})$ pos. semidefn. for all $\underline{x}$ $\Rightarrow$ any stationary pt is global min
f concave $\Leftrightarrow$ " neg. semidefn. —"— $\Rightarrow$ any stationary pt is global max

Special case: f is a polynomial fn. of degree ≤ 2 $\quad f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

Then:

i) $f'(\underline{x}) = \begin{pmatrix} f'_{x_1} \\ \vdots \\ f'_{x_n} \end{pmatrix} = 2A \cdot \underline{x} + B^T \overset{\text{Stat. pts}}{=} \underline{0} \Leftrightarrow 2A \cdot \underline{x} = -B^T$

ii) $H(f) = 2A$

① Constrained optimization : Equality constraints → Lagrange problem

Case of equality constraints

Max/min $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ when

$$\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

$f(\underline{x})$: objective fn. — $g_1, \ldots, g_m$: fn's — $a_1, \ldots, a_m$: const. — the system: constraints

Candidate pts: Method of Lagrange multipliers

Lagrangian: $L(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m)$ ($\lambda_1, \ldots, \lambda_m$: Lagrange multipliers)

$$= f(x_1, \ldots, x_n) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \lambda_2 (g_2(\underline{x}) - a_2) - \ldots - \lambda_m (g_m(\underline{x}) - a_m)$$

Candidates for max/min: Stationary pts of L

$$\left.\begin{aligned} L'_{x_1} &= f'_{x_1} - \lambda_1 \cdot (g_1)'_{x_1} - \ldots - \lambda_m \cdot (g_m)'_{x_1} = 0 \\ &\vdots \\ L'_{x_n} &= f'_{x_n} - \lambda_1 \cdot (g_1)'_{x_n} - \ldots - \lambda_m \cdot (g_m)'_{x_n} = 0 \end{aligned}\right\} \text{FOC}$$

$$\left.\begin{aligned} L'_{\lambda_1} &= -(g_1(\underline{x}) - a_1) = 0 \iff g_1(\underline{x}) = a_1 \\ &\vdots \\ L'_{\lambda_m} &= -(g_m(\underline{x}) - a_m) = 0 \iff g_m(\underline{x}) = a_m \end{aligned}\right\} \text{C}$$

FOC + C = Lagrange conditions

Ex: max/min $f(x,y,z) = x+y+z$ when $4x^2+9y^2+z^2=36$

$$L(x,y,z;\lambda) = x+y+z-\lambda(4x^2+9y^2+z^2-36)$$

FOC:
$$\begin{cases} L'_x = 1-\lambda\cdot 8x = 0 \\ L'_y = 1-\lambda\cdot 18y = 0 \\ L'_z = 1-\lambda\cdot 2z = 0 \\ 4x^2+9y^2+z^2=36 \end{cases}$$

$1 = 8\lambda x$ $\quad x = \frac{1}{8\lambda} = \frac{1}{4}\cdot\frac{1}{2\lambda} = \pm\frac{1}{4}\cdot\frac{36}{7} = \pm\frac{9}{7}$

$1 = 18\lambda y$ $\quad y = \frac{1}{18\lambda} = \frac{1}{9}\cdot\frac{1}{2\lambda} = \pm\frac{1}{9}\cdot\frac{36}{7} = \pm\frac{4}{7}$

$1 = 2\lambda z$ $\quad z = \frac{1}{2\lambda} = \pm\frac{36}{7}$

c:
$$4\left(\frac{1}{8\lambda}\right)^2 + 9\cdot\left(\frac{1}{18\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 36$$

$$\left(\frac{1}{2\lambda}\right)^2\cdot\left(\frac{4}{4^2}+\frac{9}{9^2}+1\right) = 36 \qquad \left(\frac{1}{2\lambda}\right)^2\cdot\frac{49}{36} = 36$$

$$\frac{1\cdot 9}{4\cdot 9}+\frac{1\cdot 4}{9\cdot 4}+1\cdot\frac{36}{36} = 49/36$$

$$\left(\frac{1}{2\lambda}\right)^2 = \frac{36\cdot 36}{49}$$

$$\frac{1}{2\lambda} = \pm\sqrt{\frac{36^2}{49}} = \pm\frac{36}{7}$$

$$2\lambda = \pm\frac{7}{36} \qquad \lambda = \pm\frac{7}{72}$$

$\lambda = \frac{7}{72}$: $x = \frac{9}{7}$ $y = \frac{4}{7}$ $z = \frac{36}{7}$

$\lambda = -\frac{7}{72}$: $x = -\frac{9}{7}$ $y = -\frac{4}{7}$ $z = -\frac{36}{7}$

Candidate pts: $(x,y,z;\lambda) = (9/7, 4/7, 36/7; 7/72)$, $f = \frac{49}{7} = 7$

$(-9/7, -4/7, -36/7; -7/72)$ $f = -\frac{49}{7} = -7$

Extreme Value Theorem: (EVT)

If f is a cont. fn. on a compact set D, then f has a max and a min on D.

Obj. fn: $f(x_1, \dots, x_n)$ — f is cont. in all our cases

Set D: D is the set of all pts $(x_1, \dots, x_n)$ such that all constraints are satisfied

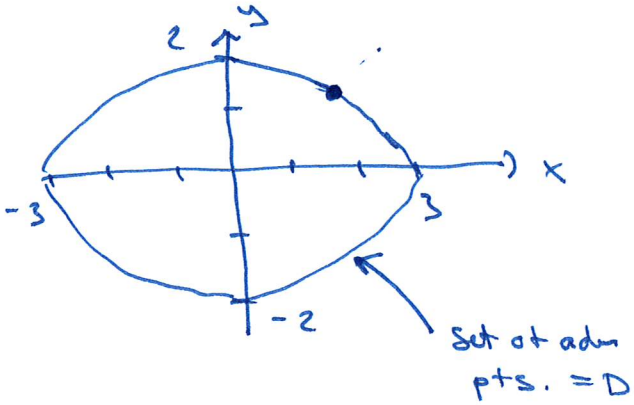
the set of adm. pts $= D$

D is compact if it is closed and bounded

Ex: $D: \boxed{4x^2 + 9y^2 = 36}$ 1:36

$$\frac{4x^2}{36} + \frac{9y^2}{36} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



Defn:

A boundary pt of D is a pt where there are pts in D and points outside of D infinitely close. The set D is closed if all boundary pts of D are included in D.

Fact: $=$ gives boundary pts; $<$ $>$ " not boundary pts } set defined by $=$, $\leq$ or $\geq$ are closed

$\Rightarrow$ In a Lagrange problem, D is closed.

The set D is bounded if there are constants $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that

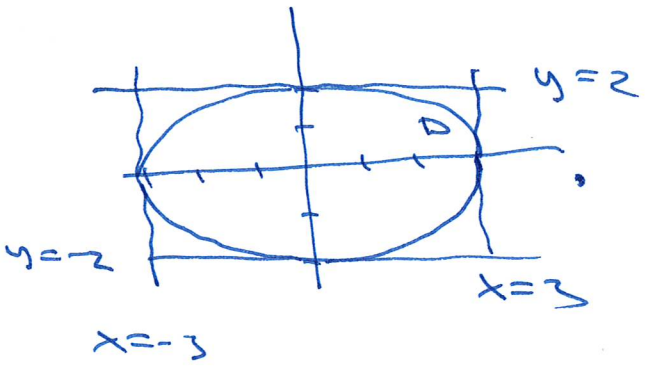
$$a_1 \le x_1 \le b_1$$
$$a_2 \le x_2 \le b_2$$
$$\vdots$$
$$a_n \le x_n \le b_n$$

for all pts $(x_1, x_2, \dots, x_n)$ in D.

D: $4x^2 + 9y^2 = 36$

$-3 \le x \le 3$

$-2 \le y \le 2$



$9y^2 \ge 0 \Rightarrow 4x^2 \le 36$

$x^2 \le 9$

$-3 \le x \le 3$

$9y^2 \le 36 \quad y^2 \le 4$

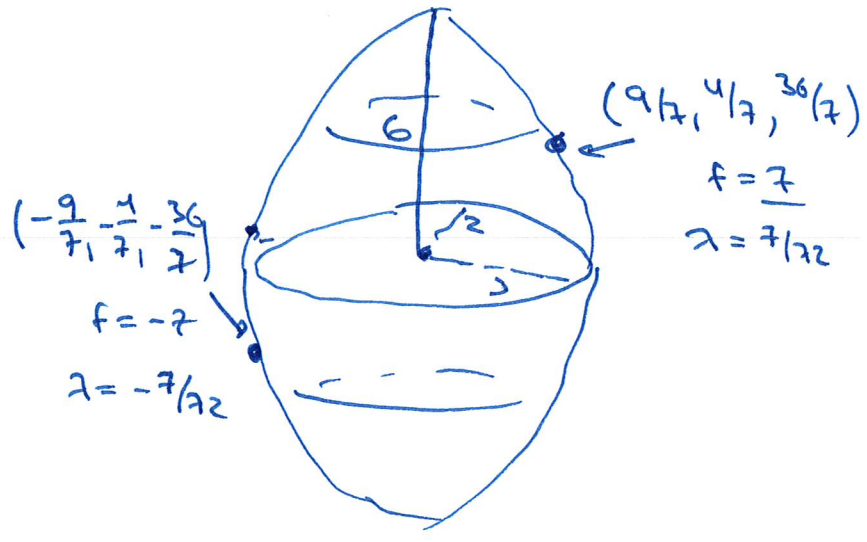
$-2 \le y \le 2$

Ex: $4x^2 + 9y^2 + z^2 = 36$ is compact

$-3 \le x \le 3$

$-2 \le y \le 2$

$-6 \le z \le 6$



Concl: $f_{max} = 7$

$f_{min} = -7$

② Second order conditions

Ex: max/min $f = x+y+z$ s.t. $4x^2+9y^2+z^2 = 36$

Candidate for max: $(x,y,z;\lambda) = (9/7, 4/7, 36/7; 7/72)$

— " — min: $(x,y,z;\lambda) = (-9/7, -4/7, -36/7; -7/72)$

Second order condition: (SOC)

If $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$ satisfies FOC+C in a Lagrange problem, then we consider

$$h(x_1, x_2, \dots x_n) = L(x_1, x_2, \dots x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$$

We have: h concave $\Rightarrow \underline{x}^*$ is max, $f_{max} = f(\underline{x}^*)$

h convex $\Rightarrow \underline{x}^*$ is min, $f_{min} = f(\underline{x}^*)$

Ex (cont'd): $L = x+y+z - \lambda(4x^2+9y^2+z^2-36)$

$(x^*, y^*, z^*; \lambda^*) = (9/7, 4/7, 36/7; 7/72)$ $f = 7$:

$$h(x,y,z) = x+y+z - \frac{7}{72}(4x^2+9y^2+z^2-36) \quad \text{Concave?}$$

$H(h) = 2A$ $= \underline{x}^T A \underline{x} + B\underline{x} + C$

$B = (1\ 1\ 1)$, $C = \frac{7}{2}$

$$= -\frac{7}{36}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -7/9 & 0 & 0 \\ 0 & -7/4 & 0 \\ 0 & 0 & -7/36 \end{pmatrix} \quad \begin{matrix} \lambda_1 = -7/9 \\ \lambda_2 = -7/4 \\ \lambda_3 = -7/36 \end{matrix}$$

$A = -\frac{7}{72}\cdot\begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$H(h)$ negative defn. $\Rightarrow H(h)$ negative semidefn.

$\Rightarrow h$ concave fn. $\underset{\text{SOC}}{\Rightarrow}$ $(9/7, 4/7, 36/7; 7/72)$ max, $f_{max} = \underline{7}$

Ex: (cont'd, for min) $L = x+y+z-\lambda(4x^2+9y^2+z^2-36)$

$(-9/7, -4/7, -36/7; -7/72)$ $f = -7$

$h(x,y,z) = x+y+z+\frac{7}{72}(4x^2+9y^2+z^2-36)$ convex?

$$h'_x = 1+\frac{7\cdot 4}{72}\cdot 2x = \underline{1+\frac{7}{9}x}$$

$$h'_y = 1+\frac{7\cdot 9}{72}\cdot 2y = \underline{1+\frac{7}{4}y}$$

$$h'_z = 1+\frac{7}{72}\cdot 2z = \underline{1+\frac{7}{36}z}$$

$$H(h) = \begin{pmatrix} 7/9 & 0 & 0 \\ 0 & 7/4 & 0 \\ 0 & 0 & 7/36 \end{pmatrix} \quad \lambda_1 = 7/9,\ \lambda_2 = 7/4,\ \lambda_3 = 7/36$$

$H(h)$ pos. def (pos. semidef.) $\Rightarrow h$ convex $\Rightarrow$ SOC $(-9/7, -4/7, -36/7)$ is min pt.

$f_{min} = \underline{\underline{-7}}$

Thm (Necessary conditions)

If $(x_1^*, x_2^*, \dots, x_n^*)$ is a max or a min in a Lagrange problem, then either

i) there are $\lambda_1^*, \dots, \lambda_m^*$ such that $(x_1^*, \dots, x_n^*; \lambda_1^*, \dots, \lambda_m^*)$ satisfies FOC + C (ordinary candidate points)

or

ii) $(x_1^*, \dots, x_n^*)$ is an admissible point where NDCQ = non-degenerate constraint qualification fails (exceptional candidate points)

NDCQ:

$$\operatorname{rk} \begin{pmatrix} (g_1)'_{x_1} & (g_1)'_{x_2} & \cdots & (g_1)'_{x_n} \\ (g_2)'_{x_1} & (g_2)'_{x_2} & \cdots & (g_2)'_{x_n} \\ \vdots & \vdots & & \vdots \\ (g_m)'_{x_1} & (g_m)'_{x_2} & \cdots & (g_m)'_{x_n} \end{pmatrix} = m$$

Ex: $4x^2 + 9y^2 + z^2 = 36$ (C)

NDCQ: $\operatorname{rk}(8x \;\; 18y \;\; 2z) = 1$

NDCQ fails: $\operatorname{rk}(8x \;\; 18y \;\; 2z) = 0$

$8x = 0 \quad 18y = 0 \quad 2z = 0$

$x = 0 \quad y = 0 \quad z = 0 \qquad g(0,0,0) = 0 \neq 36$

not adm.