	Plan	Textbook
1	Kuhn-Tucker problems	[E] 6.5
2	Examples: Minimum variance portfolios	[E] 6.7
	Textbook problems	[E] 6.3cd, 6.5 - 6.7, 6.10

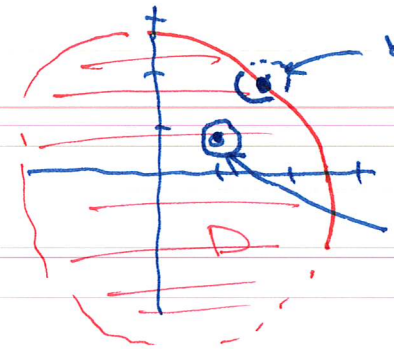
Constrained optimization
- Lagrange problems: equality constraints $=$
- Kuhn-Tucker problems: closed inequality constraints $\leqslant$ $\geqslant$

① Kuhn-Tucker problems

Def. A Kuhn-Tucker problem in standard form is a constrained optimization problem of the form

$$\max f(\underline{x}) = f(x_1, x_2, \dots, x_n) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) = g_1(x_1, \dots x_n) \leqslant a_1 \\ g_2(\underline{x}) \leqslant a_2 \\ \vdots \\ g_m(\underline{x}) \leqslant a_m \end{cases}$$

Ex: $\max f(x,y) = xy$ when $x^2+y^2 \leqslant 8$

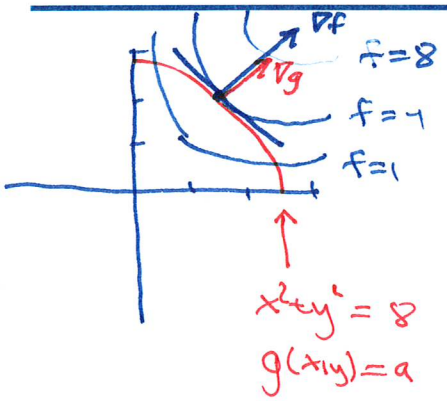


boundary pt of D $x^2+y^2=8$
constraint is binding
Cand pt: $h'_x = h'_y = 0$ $\boxed{\lambda \geqslant 0}$

interior pt of D $x^2+y^2<8$
constraint is non-binding
Cand pt: $f'_x = f'_y = 0$ $\boxed{\lambda = 0}$

D: $x^2+y^2 \leqslant 8$
closed and bounded
= compact

$L = xy - \lambda(x^2+y^2-8)$
$L'_x = y - \lambda \cdot 2x = 0$
$L'_y = x - \lambda \cdot 2y = 0$



$f(x,y)=c$ level curves

$xy=c$

red and blue curve meet at a tangent

$$\Updownarrow$$

$$\boxed{\nabla f = \lambda \cdot \nabla g}$$

$$\left.\begin{aligned} f'_x &= \lambda \cdot g'_x \\ f'_y &= \lambda \cdot g'_y \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} L'_x &= 0 \\ L'_y &= 0 \end{aligned}\right.$$

$\nabla f = \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$ gradient of f = vector that points in the direction where f grows fastest, and is orthogonal to the tangent

$\nabla g = \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$

Kuhn-Tucker formulation: $\max f(\underline{x})$ s.t. $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

$$L(\underline{x};\underline{\lambda}) = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \lambda_2 (g_2(\underline{x}) - a_2) - \dots - \lambda_m (g_m(\underline{x}) - a_m)$$

$L'_{x_1} = 0$	$g_1(\underline{x}) \le a_1$	$\lambda_1 \ge 0$, $\lambda_1 = 0$ if $g_1(\underline{x}) < a_1$
$L'_{x_2} = 0$	$g_2(\underline{x}) \le a_2$	$\lambda_2 \ge 0$, $\lambda_2 = 0$ if $g_2(\underline{x}) < a_2$
$\vdots$	$\vdots$	$\vdots$
$L'_{x_n} = 0$	$g_m(\underline{x}) \le a_m$	$\lambda_m \ge 0$, $\lambda_m = 0$ if $g_m(\underline{x}) < a_m$
FOC	C	CSC = complimentary slackness conditions

Kuhn-Tucker conditions: FOC + C + CSC

Thm:

If $\underline{x}^*$ is a max in the KT-problem, then either

i) $(\underline{x}^*;\underline{\lambda}^*)$ satisfies FOC+C+CSC for some $\underline{\lambda}^*$

or

ii) $\underline{x}^*$ is an adm. pt where NDCQ fails

Ex: max $f = xy$ s.t. $x^2 + y^2 \leq 8$

$D: x^2 + y^2 \leq 8$ compact ⇒ EVT there is a max

$L = xy - \lambda(x^2 + y^2 - 8)$

$L'_x = y - \lambda \cdot 2x = 0$ $\quad x^2 + y^2 \leq 8$ $\quad \lambda \geq 0$

$L'_y = x - \lambda \cdot 2y = 0$ $\quad \lambda = 0$ if $x^2 + y^2 < 8$

FOC (=) CSC

$\lambda \geq 0$ $x^2 + y^2 = 8$

$y - \lambda \cdot 2x = 0$

$\Rightarrow y = 2\lambda x$

$x - 2x \cdot (2\lambda x) = 0$

$x(1 - 4\lambda^2) = 0$

$x = 0$ or $\lambda^2 = 1/4$

$x = 0$	$\lambda = 1/2$	$\lambda = -1/2$
$y \neq 0$	$y = x$	$\lambda \geq 0$
$x^2 + y^2 \neq 8$	$2x^2 = 8$	
	$x^2 = 4$	
	$x = \pm 2$	

$(2, 2; 1/2)$ $f = 4$

$(-2, -2; 1/2)$ $f = 4$

$x^2 + y^2 < 8$ **$\lambda = 0$**

$\lambda = 0$

$y = 0$

$x = 0$

$0^2 + 0^2 = 0 < 8$ ok

Cand: $(0, 0; 0)$ $f = 0$

Second order condition:

If $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC + C + CSC, then: $h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$ concave $\Rightarrow \underline{x}^*$ is max

Ex: $h(x, y) = L(x, y; 1/2) = xy - \frac{1}{2}(x^2 + y^2 - 8)$

$h'_x = y - x$

$h'_y = x - y$

$Hh(x) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $D_1 = -1$ $D_2 = 0$ ⇒ neg. semidefin.

RKC ⇒ h concave ⇒ SOC $f_{max} = 4$ at $(2, 2)$, $(-2, -2)$

NDCQ in the Kuhn-Tucker case:

Let $\underline{x}^*$ be an adm pt: $\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$ (B =) (NB <)

NDCQ: $\operatorname{rk}\begin{pmatrix} (g_i)'_{x_1} & \cdots & (g_i)'_{x_n} \\ \vdots & & \end{pmatrix}$ is maximal

↑ include only rows corresp^ing to binding constraints

Ex: $x^2+y^2 \leq 8$ — B: $x^2+y^2=8$ $\operatorname{rk}(2x\ 2y)=1$, $x=0, y=0$ $0^2+0^2 \neq 8$ (ok); NB: $x^2+y^2<8$ no condition (ok)

Ex: $\min f(x,y,z) = x^2+y^2+z^2$ when $\begin{cases} x+y+z \leq 4 \\ x-y+3z \geq 9 \end{cases}$ KT prob. not in std. form

graph of f

graph of $-f$

$x-y+3z \geq 9 \quad | \cdot (-1)$

$-x+y-3z \leq -9$

$\max -f(x,y,z) = -x^2-y^2-z^2$ when $\begin{cases} x+y+z \leq 4 \\ x+y-3z \leq -9 \end{cases}$ KT prob. in std form

$L = -x^2-y^2-z^2 - \lambda_1(x+y+z-4) - \lambda_2(-x+y-3z+9)$

$L'_x = -2x-\lambda_1+\lambda_2 = 0$

$L'_y = -2y-\lambda_1-\lambda_2 = 0$

$L'_z = -2z-\lambda_1+3\lambda_2 = 0$

$x = \frac{-\lambda_1+\lambda_2}{2}$

$y = \frac{-\lambda_1-\lambda_2}{2}$

$z = \frac{-\lambda_1+3\lambda_2}{2}$

$x+y+z \leq 4$ $\quad \lambda_1 \geq 0$ $\lambda_1=0$ if $x+y+z<4$

$-x+y-3z \leq -9$ $\quad \lambda_2 \geq 0$ $\lambda_2=0$ if $-x+y-3z<-9$

Ex: $\min f(x,y,z,w) = 3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 7z^2 - 4w^2$ when $x+y+z+w \le 6$

$= \underline{x}^T A \underline{x}$ $\qquad$ $B\underline{x}$

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix} \qquad B = (1\ 1\ 1\ 1)$$

Std. form: $\max -f(\underline{x}) = -\underline{x}^T A \underline{x}$ when $B\underline{x} \le 6$

$L = -\underline{x}^T A \underline{x} - \lambda(B\underline{x} - 6)$

FOC: $L'(\underline{x}) = -2A\underline{x} - \lambda(B^T) = \underline{0}$

C: $B\underline{x} \le 6$

CSC: $\lambda \ge 0$, $\lambda = 0$ if $B\underline{x} < 6$

FOC: $-2A\underline{x} - \lambda \cdot B^T = \underline{0}$

$-2A\underline{x} = \lambda \cdot B^T \quad | : (-2)$

$t = -\lambda/2$

$$\begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix} \underline{x} = A\underline{x} = \frac{\lambda}{-2} \cdot B^T = -\frac{\lambda}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda/2 \\ -\lambda/2 \\ -\lambda/2 \\ -\lambda/2 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 1 & 4 & -1 & t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \sim \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 2 & 1 & t \\ 3 & 1 & 4 & -1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \begin{matrix} \\ \cdot(-3) \\ \\ \cdot(-4) \\ \cdot(1) \end{matrix} \sim \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 2 & 1 & t \\ 0 & \textcircled{-2} & -2 & -4 & -2t \\ 0 & -2 & -1 & -4 & -3t \\ 0 & 2 & 2 & 5 & 2t \end{array}\right) \begin{matrix} \\ \\ \\ \cdot(-1) \end{matrix}$$

$$\sim \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 2 & 1 & t \\ 0 & \textcircled{-2} & -2 & -4 & -2t \\ 0 & 0 & \textcircled{1} & 0 & -t \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right)$$

$x + 2t + 2(-t) + 0 = t \Rightarrow x = t$

$-2y - 2(-t) - 4(0) = -2t \Rightarrow -2y = -4t \quad y = 2t$

$z = -t$

$w = 0$

FOC: $(x,y,z,w) = (t, 2t, -t, 0)$ $\quad t = -\lambda/2$

$Bx = 6$	$Bx < 6$
$\lambda \geq 0$	$\lambda = 0 \Rightarrow t = 0$
$x+y+z+w = 6$	$(x,y,z,w) = (0,0,0,0)$
$t + 2t - t + 0 = 6$	$(0,0,0,0;0)$
$2t = 6$	$x+y+z+w = 0 < 6$ (ok)
$t = 3$	
$t = -\frac{\lambda}{2} = 3$	Check with SOC:
$\lambda = -6$	$h = L(x;0) = -x^T A x - 0(Bx - 6)$
$(3, 6, -3, 0; -6)$	$= -x^T A x$
$\lambda = -6 < 0$	$H(h) = -2A$

$$A = \begin{pmatrix} 3 & 4 & 4 & -1 \\ 1 & 1 & 2 & +1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix} \qquad D_1 = 3,\ D_2 = 2,\ D_3 = 2,\ D_4 = 2$$

$D_3 = 4(-2) - 2(2) + 7 \cdot 2 = 2$

$D_4 = 2$

h concave $\Leftarrow$ $H(h) = -2A$ $\Leftrightarrow$ A pos. semidefin.

SOC ✓ neg. semidefin.

$(0,0,0,0)$ is max for $-f$ = min for f

Concl: $f_{min} = 0$ at $(0,0,0,0)$ with $\lambda = 0$