	Plan	Textbook
1	Envelope Theorems	[E] 5.7, 6.6
2	Examples	
	Textbook problems	[E] 5.13 - 5.14, 6.8 - 6.9
	Exam problems	Final exam 01/2018 Question 3

Plenary session 3
Thu Oct 30th
at 16-19 in A1-040

Unconstrained optimization : $\max_{x} f(x)$

- Lagrange problems : max/min $f(x)$ s.t. ... (equality c.) $=$
- Kuhn-Tucker problems: max/min $f(x)$ s.t. ... (closed ineq. c.) $\leq$ $\geq$

① Envelope Theorems

Ex: $\max f(x,y,z,w) = xw - yz$ s.t. $\begin{cases} x^2+4y^2 \leq 4 \\ 4z^2+9w^2 \leq 36 \end{cases}$ KT problem in std form

$$L = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$

FOC:
$$L'_x = w - \lambda_1 \cdot 2x = 0$$
$$L'_y = -z - \lambda_1 \cdot 8y = 0$$
$$L'_z = -y - \lambda_2 \cdot 8z = 0$$
$$L'_w = x - \lambda_2 \cdot 18w = 0$$

C: $x^2+4y^2 \leq 4$
$4z^2+9w^2 \leq 36$

CSC: $\lambda_1 \geq 0$
$\lambda_1 = 0$ if $x^2+4y^2 < 4$
$\lambda_2 \geq 0$
$\lambda_2 = 0$ if $4z^2+9w^2 < 36$

(1) and (4)
$w = 2\lambda_1 x$: $x - 18\lambda_2 \cdot (2\lambda_1 x) = 0$
$x(1 - 36\lambda_1\lambda_2) = 0$
$x=0$, $w=0$ or $\lambda_1\lambda_2 = 1/36$

(2) and (3)
$z = -8\lambda_1 y$: $-y - 8\lambda_2 \cdot (-8\lambda_1 y) = 0$
$y(-1 + 64\lambda_1\lambda_2) = 0$
$y=0$, $z=0$ or $\lambda_1\lambda_2 = 1/64$

Cases:

(A) $x=0, y=0$: $w=0, z=0$
C: $0 < 4$, $0 < 36$ $\lambda_1 = 0$, $\lambda_2 = 0$ Cand pts: $f=0$ $(0,0,0,0; 0,0)$

(B) $x=0, \lambda_1\lambda_2 = 1/64$: $w=0$
$\lambda_1 > 0$ $4y^2=4$ $y=\pm 1$ $\lambda_1 = \frac{-z}{8y} = \frac{3}{8}$
$\lambda_2 > 0$ $4z^2=36$ $z=\pm 3$ $\lambda_2 = 1/24$

(C) $\lambda_1\lambda_2 = 1/36$, $y=0$
Cand pts: $f=3$
$(0,1,-3,0; 3/8, 1/24)$
$(0,-1,3,0; 3/8, 1/24)$

(c) $\lambda_1\lambda_2 = 1/36$, $y=0$: $z=0$

$\lambda_1 > 0$: $x^2 = 4$ $x = \pm 2$ $\lambda_1 = \frac{w}{2x} = \frac{1}{2}$

$\lambda_2 > 0$: $9w^2 = 36$ $w = \pm 2$ $\lambda_2 = \frac{1}{18}$

Cand pts:

$(2,0,0,2;\ 1/2,\ 1/18)$ $f=4$

$(-2,0,0,-2;\ 1/2,\ 1/18)$

Highest f-values = best cand. pt.

Test: SOC

$$h(x,y,z,w) = xw - yz - \tfrac{1}{2}(x^2+4y^2-4) - \tfrac{1}{18}(4z^2+9w^2-36) = \underline{x}^T A \underline{x} + 2 + 2$$

$$H(h) = 2A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -4 & -1 & 0 \\ 0 & -1 & -4/9 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$D_1 = -1$

$D_2 = 4$

$D_3 = -1\left(16/9 - 1\right) = -7/9$

$D_4 = 0$ since $R_4 = -R_1 \Rightarrow \rho_4 = 3$

ERC $\Rightarrow$ neg. semidefin.

h concave $\overset{\text{SOC}}{\Rightarrow}$ $f_{max} = 4 = f^*$

$x^* = 2$ $y^* = 0$ $z^* = 0$ $w^* = 2$ $\lambda_1^* = \frac{1}{2}$ $\lambda_2^* = \frac{1}{18}$

or

$x^* = -2$, $y^* = 0$, $z^* = 0$, $w^* = -2$ $\lambda_1^* = \frac{1}{2}$ $\lambda_2^* = \frac{1}{18}$

What if: max $f = xw - yz$ s.t. $x^2 + 4y^2 \le 4$, $4z^2 + 9w^2 \le 45$

Introduce parameter a: Max $f = xw - yz$ s.t. $\begin{cases} x^2 + 4y^2 \le 4 \\ 4z^2 + 9w^2 \le a \end{cases}$

Notation: $f^*(a)$ max value ← optimal value function

know: $f^*(36) = 4$

$x^*(a)$, $y^*(a)$, $z^*(a)$, $w^*(a)$ } max pt

know:

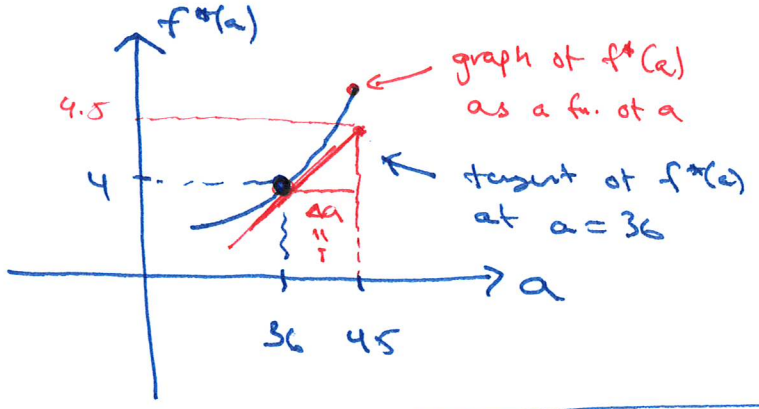
$x^*(36) = (2,0,0,2),\ (-2,0,0,-2)$

$\lambda_1^*(a)$, $\lambda_2^*(a)$ } corr. Lagrange multipliers

know:

$\lambda_1^*(36) = 1/2$, $\lambda_2^*(36) = 1/18$

Want to find: $f^*(45)$



Slope of tangent: $\frac{df^*(a)}{da}$

Envelope thm: (K.T. core)

$$\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \underline{\lambda}^*(a))$$

Explanation: $L = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-a)$

$= \dots\dots + \lambda_2 \cdot a$

$L'_a = \lambda_2 \Rightarrow \frac{df^*(a)}{da} = \lambda_2^*(a) = \frac{1}{18}$ (a=36)

$f^*(45) \simeq f^*(36) + \Delta a \cdot \frac{df^*(a)}{da} = 4 + 9 \cdot \frac{1}{18} = 4.5$

($f^*(36) = 4$; $\Delta a = 45-36 = 9$; $\frac{df^*(a)}{da} = 1/18$)

What about: $\max f = xw - 2yz$ s.t. $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le 36 \end{cases}$ $a=1$: old pb.

$\max f = xw - ayz$ s.t. $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le 36 \end{cases}$ $a=2$: new pb.

$\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \lambda^*(a)) = -0.0 = 0$

$L = xw - ayz - \lambda_1 \dots - \lambda_2 \dots$

$L'_a = -yz$

What about: $\max_{w,a} xw - yt$ s.t. $\begin{cases} \leq \\ = \\ \geq \end{cases}$

$$\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \underline{\lambda}^*(a)) = x^*(a) \cdot w^*(a) = 4$$

$L = axw - yt - \dots$

$L'_a = xw$

$$f^*(1.2) \cong f^*(1) + \Delta a \cdot \frac{df^*(a)}{da} \cong 4.8$$

$f^*(1)$ = 4, Δa = $1.2 - 1$ = 0.2, $\frac{df^*(a)}{da}$ = 4

Envelope thm:

Lagrange/KT-case: $\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \lambda^*(a))$

Unconstrained case: $\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$

Interpretation of the Lagrange multipliers

$\max f(\underline{x})$ s.t.

$g_1(\underline{x}) = a_1$ (or $\leq$)

$g_2(\underline{x}) = a_2$ (or $\leq$) ← Put $a = a_2$

$\vdots$

$g_m(\underline{x}) = a_m$ (or $\leq$)

$\rightsquigarrow L = f(\underline{x}) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \lambda_2 (g_2(\underline{x}) - a) - \dots$

$= \lambda_2 \cdot a + \dots \Rightarrow L'_a = \lambda_2$

Fact: When a is the constant in the i'th constraint, then $\frac{df^*(a)}{da} = \lambda_i$

Interpretation of the Lagrange multipliers at a maximum

λ_i = marginal change in the optimal value per unit change in the constant a_i in the i'th constraint.

Ex: $\max f(x) = 1 + 2x - x^2$

$f'(x) = 2 - 2x = 0$ $\quad f''(x) = -2 < 0$

Stat pt: $x = 1$ $\quad$ f concave

$\Rightarrow f_{max} = f(1) = 2$ $\quad x^* = 1$ max pt.

$\quad\quad\quad

$f(x) = 1 + ax - x^2$

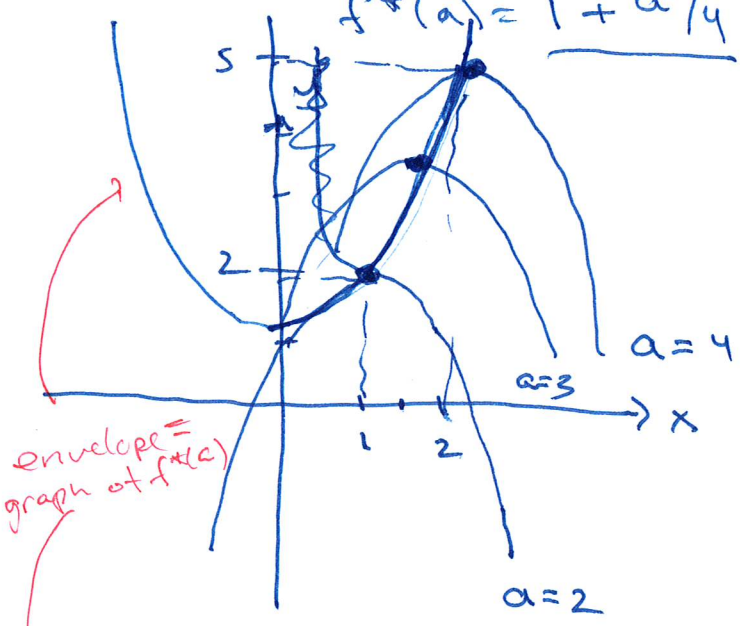
$f'(x) = a - 2x = 0$ $\quad f'' = -2 < 0 \Rightarrow x^*(a) = a/2$

Stat. pt: $x = a/2$ $\quad$ f concave $\quad$ $f^*(a) = 1 + a \cdot a/2 - (a/2)^2$

$= 1 + a^2/2 - a^2/4$

Concl: $x^*(a) = a/2$ max pt. $\quad$ $= 1 + \frac{a^2}{4}$

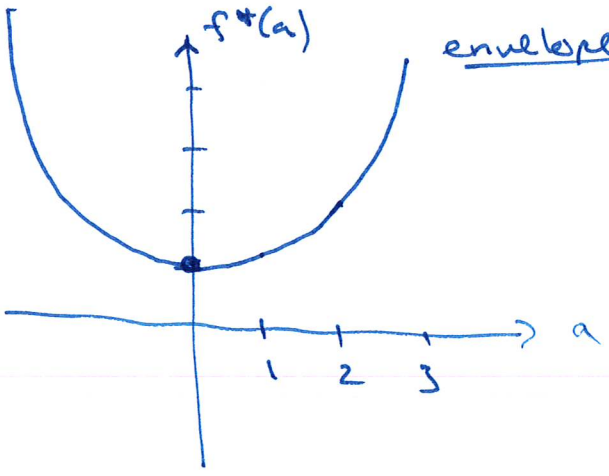
$f^*(a) = 1 + a^2/4$ max value



$a = 2$: $x^*(2) = 1$ $\quad f^*(2) = 2$

$a = 3$: $x^*(a) = 1.5$ $\quad f^*(a) = 3.25$

$a = 4$: $x^*(a) = 2$ $\quad f^*(a) = 5$



envelope

$f^*(a) = 1 + a^2/4$ $\quad f^*(3) = 3.25$

$\frac{df^*(a)}{da} = 0 + 2a/4 = a/2$