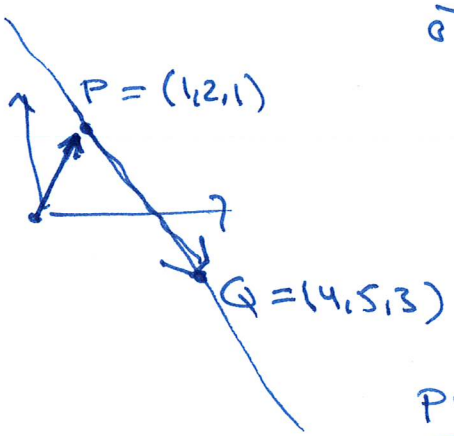
	Plan	
1	Key Problems:	2.2, 2.4, 3.4c, 3.5, 4.1d, 4.2, 4.3, 4.4, 4.5
2	Textbook Problems:	1.14, 2.14
3	Exam Problems:	Midterm 01/2020 Q4, 10/2021 Q3, 01/2023 Q2-3

① Key problems

2.2



$\vec{OP} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ $\vec{PQ} = \begin{pmatrix}3\\3\\2\end{pmatrix}$

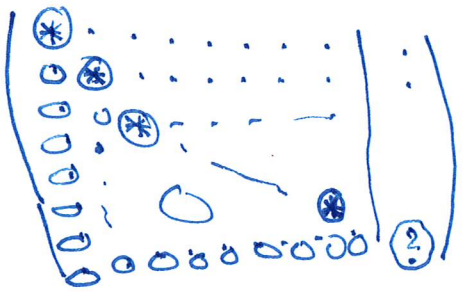
$\begin{pmatrix}1\\2\\1\end{pmatrix} + t\cdot\begin{pmatrix}3\\3\\2\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ $\quad x = 1+3t,\ y = 2+3t,\ z = 1+2t$

Plane: $x - y + z = 6$

$(1+3t) - (2+3t) + (1+2t) = 6$

$2t = 6 \quad t = 3 \quad (x,y,z) = (10,11,7)$

2.4 $Ax = b$



a) $\dim \text{Null}(A) = 1$ $\dim \text{Col}(A) = 7$

b) Int. many solutions $(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) \rightarrow (0\ 0\ \ldots\ 0\ ?\ 1\ 1)$

c) $? = 0$: one free var, int. many sol's $\quad ? \neq 0$: no sol'n's

d) $Ax = 0$ and $x_1 + x_2 + \cdots + x_8 = 1$: $? = 0$ no sol'n's $\quad ? \neq 0$: one unique sol'n.

3.4 c $A = \begin{pmatrix} 1 & a & b \\ a & b & c \end{pmatrix}$

$M_{12,12} = b - a^2 = 0$

$M_{12,13} = c - ab = 0$

$M_{12,23} = ac - b^2 = 0$

$\text{rk } A < 2$ if (i.e $\text{rk } A = 1$)

$b = a^2 \quad c = ab = a \cdot a^2 = a^3$

$ac - b^2 = a \cdot a^3 - (a^2)^2 = 0$

$b = a^2$, $c = a^3$, a free : $\text{rk } A = 1$

otherwise : $\text{rk } A = 2$

3.5 $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & t \end{array}\right) \begin{matrix} t \\ 2 \\ 4 \end{matrix}$

$M_{12,12} = 3 - 2 = 1 \neq 0$

$M_{123,123} = +1(14) - 1 \cdot 10 + 2 \cdot (-2) = 0$

$M_{123,124} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & 5 & t \end{vmatrix} = +1(3t+10) - 1(2t+8) + 1(-2)$

$= 3t + 10 - 2t - 8 - 2 = t$

$t \neq 0$: Inf. many solns, z free

$t = 0$: no soln's

$M_{123,125} = \begin{vmatrix} 1 & 1 & t=0 \\ 2 & 3 & 2 \\ 4 & 5 & 4 \end{vmatrix} = 1 \cdot 2 - 1 \cdot 0 = 2 \neq 0$

$t = 1$: Inf. many soln's, z free

know that $\det = 1 \neq 0$

$x + y + 2z + w = 1$

$2x + 3y - z - 2w = 2$

$4x + 5y + 3z + w = 4$

$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 1 - 2z \\ 2 + z \\ 4 - 3z \end{pmatrix}$

$x + y + w = 1 - 2z$

$2x + 3y - 2w = 2 + z$

$4x + 5y + w = 4 - 3z$

$\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & 5 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 - 2z \\ 2 + z \\ 4 - 3z \end{pmatrix}$

$= \frac{1}{1} \begin{pmatrix} 13 & -10 & -2 \\ 4 & -3 & -1 \\ -5 & 4 & 1 \end{pmatrix}^T \cdot \begin{pmatrix} 1 - 2z \\ 2 + z \\ 4 - 3z \end{pmatrix} = \begin{pmatrix} 13 & 4 & -5 \\ -10 & -3 & 4 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 - 2z \\ 2 + z \\ 4 - 3z \end{pmatrix} = \begin{pmatrix} 1 - 7z \\ 5z \\ 0 \end{pmatrix}$

4.1 d) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ $\quad A\underline{v} = \lambda \underline{v}$

$A\underline{v} - \lambda\underline{v} = \underline{0}$

$(A - \lambda I)\underline{v} = \underline{0}$

Eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0 \quad \leftarrow \text{char. equ.}$$

$$+ (4-\lambda) \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)\cdot(\lambda^2 - 6\lambda + 8) = 0$$

$4-\lambda = 0$ or $\lambda^2 - 6\lambda + 8 = 0$ $\qquad \lambda = 4 \quad (m=2)$

$\lambda_1 = 4$ $\qquad \lambda_2 = 4, \lambda_3 = 2$ $\qquad \lambda = 2 \quad (m=1)$

Eigenvectors:

$\lambda = 4$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad -x + z = 0$ $\quad x = z$

y, z free $\quad y = y$

$z = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix}$$

$$= y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{v_1}$ $\qquad$ $\underline{v_2}$

Base of E_4: $\{\underline{v_1}, \underline{v_2}\}$

$\lambda = 2$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x + z = 0$ $\quad x = -z$

$2y = 0$ $\quad y = 0$

z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{v_3}$

Base of E_2 $\{\underline{v_3}\}$

4.2 d) $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ Yes, it is diagonalizable.

$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

4.3

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 4 \\ 0 & 2-\lambda & 3 & 0 \\ 0 & 3 & 2-\lambda & 0 \\ 4 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$+(1-\lambda)\cdot\begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} - 4\cdot\begin{vmatrix} 0 & 2-\lambda & 3 \\ 0 & 3 & 2-\lambda \\ 4 & 0 & 0 \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\left(+(1-\lambda)\cdot\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix}\right) - 4\left(+4\cdot\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix}\right) = 0$$

$$(\lambda^2-4\lambda-5)\cdot\left((1-\lambda)^2-4\cdot 4\right) = 0$$

$\lambda^2-4\lambda-5=0$ or $\lambda^2-2\lambda-15=0$

$\lambda=5$, $\lambda=-1$ $\qquad$ $\lambda=-3$, $\lambda=+5$

$\lambda=5 \quad (m=2)$

$\lambda=-3 \quad (m=1)$

$\lambda=-1 \quad (m=1)$

Alt 1: A is symmetric $\Rightarrow$ A diagonalizable.

Alt 2: A has 4 eigenvalues (c. with mult.) (ok) and $\dim E_5 = 2$ check (ok)

$$D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$P = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \underline{v}_4 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 0 & 0 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \\ 4 & 0 & 0 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rk}(A-5I)=2$

$\Rightarrow \dim E_5 = 4-2 = 2$

A is diagonalizable

4.4 $A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$

Eigenvalues: $\lambda^2 - 1.31\lambda + 0.31 = 0$

$\lambda = 1, \quad \lambda = 0.31$

Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} -0.23 & 0.46 \\ 0.23 & -0.46 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \end{pmatrix}$

$-0.23x + 0.46y = 0$

y free

$\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ base

$-0.23x = -0.46y$

$x = 2y$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = 0.31$: $\begin{pmatrix} 0.46 & 0.46 \\ 0.23 & 0.23 \end{pmatrix}$

$0.46x + 0.46y = 0$

y free

$\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ base

$P^{-1}AP = D \quad | \cdot P$

$AP = PD \quad | \cdot P^{-1}$

$A = PDP^{-1}$

$A^m = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$

$= P \cdot D^m \cdot P^{-1}$

$= P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^m \cdot P^{-1}$

$= P \cdot \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} P^{-1}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.31 \end{pmatrix}$

$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$A^m = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1^{m=1} & 0 \\ 0 & 0.31^m \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$\xrightarrow[m \to \infty]{} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$

5. b) $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$c_1 = \operatorname{tr} A = 3+3+3 = 9$

$c_2 = 5+5+8 = 18$

$c_3 = \det A = 3\cdot 5 - 2\cdot 4 + 1\cdot 1 = 8$

$-\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 = 0$

$\quad\;\; \| \quad\quad \| \quad\quad \|$

$\quad 9 \quad\quad 18 \quad\quad 8$

$M_{12,12} = 5$

$M_{23,23} = 5$

$M_{13,13} = 8$

$$-\lambda^3 + 9\lambda^2 - 18\lambda + 8 = 0$$

$\lambda = 2$: $-8 + 36 - 36 + 8 = 0 \Rightarrow -\lambda^3 + 9\lambda^2 - 18\lambda + 8 = 0$

$(\lambda - 2)(-\lambda^2 + 7\lambda - 4) = 0$

$-\lambda^3 + 9\lambda^2 - 18\lambda + 8 : \lambda - 2 = -\lambda^2 + 7\lambda - 4$

$-\lambda^3 + 2\lambda^2$

$7\lambda^2 - 18\lambda + 8$

$7\lambda^2 - 14\lambda$

$-4\lambda + 8$

$-4\lambda + 8$

0

$\lambda = 2$ or

$\lambda = \dfrac{-7 \pm \sqrt{49 - 16}}{-2} = \dfrac{7}{2} \pm \dfrac{\sqrt{33}}{2}$

c) $A = \begin{pmatrix} 0 & 4 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

$-\lambda^3 + 0\cdot\lambda^2 - 0\cdot\lambda + 0 = 0$

$-\lambda^3 = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 0 \quad (m = 3)$

General fact:

If A is upper triangular, then eigenvalues = diagonal entries

[E] 1.14

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & q & q^2 \end{pmatrix} \begin{matrix} \\ \\ \rfloor -1 \end{matrix} \rfloor -1 \rightarrow \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & p-1 & p^2-1 \\ 0 & q-1 & q^2-1 \end{pmatrix}$$

$p=1, q=1$: $\operatorname{rk} A = 1$

$p=1, q\neq 1$: $\operatorname{rk} A = 2$

$p\neq 1, q=1$: $\operatorname{rk} A = 2$

$p\neq 1, q\neq 1$:
- $p=q$: $\operatorname{rk} A = 2$
- $p\neq q$: $\operatorname{rk} A = 3$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & p-1 & p^2-1 \\ 0 & q-1 & q^2-1 \end{pmatrix} \begin{matrix} \\ :p-1 \\ :q-1 \end{matrix} \rightarrow \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & \textcircled{1} & p+1 \\ 0 & 1 & q+1 \end{pmatrix} \rfloor -1$$

$$\rightarrow \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & \textcircled{1} & p+1 \\ 0 & 0 & q-p \end{pmatrix}$$

$p-1\neq 0$, $p^2-1 = (p-1)(p+1)$

$q-1\neq 0$, $q^2-1 = (q-1)(q+1)$