	Plan	
1	Key Problems:	5.2d, 5.4bc, 5.5af, 6.2bcd, 6.3bd, 6.4be, 6.5ab, 7.1c, 7.2b, 7.3b, 7.4cd
2	Exam Problems:	Final 11/2023 Q3

① Key problems

5.2 d) $f = xw - yz$ — indefn.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} x \\ y \\ z \\ w \end{matrix}$$

$x\ y\ z\ w$

$D_1 = 0$ $\quad \Delta_1 = 0, 0, 0, 0$ $\quad \Delta_1 \geq 0$ pos. semidf. $\quad \Delta_1 \leq 0$ neg. semidf.

$D_2 = 0$ $\quad \Delta_2 = 0, -1/4, \ldots$ $\quad \boxed{\Delta_2 \geq 0 \quad \Delta_2 \geq 0}$

$D_3 = 0$ $\quad \Delta_3 \ldots$ $\quad \Delta_3 \geq 0 \quad \Delta_3 \leq 0$

$D_4 = 1/16$ $\quad \Delta_4 = 1/16$ $\quad \boxed{\Delta_4 \geq 0 \quad \Delta_4 \geq 0}$

5.4 c) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Eigenvalues:

$\lambda = -1$ mult $= 2$

$\lambda = 2$ mult $= 1$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

E_{-1}: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad x + y + z = 0$, y, z free $\quad x = -y - z$

$$P = \begin{pmatrix} -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

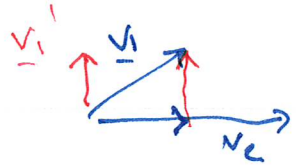
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$= v_1$ $\quad = v_2$

$v_1 \cdot v_2 = 1$

$$w_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



$v_1 = \text{proj}_{v_2} v_1 + v_1'$

$v_1' = v_1 - \text{proj}_{v_2}(v_1)$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{v_1 \cdot v_2}{v_2 \cdot v_2} \cdot v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

E_2: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \cdot 2 \to \begin{pmatrix} \textcircled{1} & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{matrix} \\ +2 \\ -1 \end{matrix} \to \begin{pmatrix} \textcircled{1} & -2 & 1 \\ 0 & \textcircled{-3} & 3 \\ 0 & 3 & -3 \end{pmatrix}$

$x - 2y + z = 0$ $\quad x = z$

$-3y + 3z = 0$ $\quad y = z$

z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\qquad w_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$= v_3$

$x^T A x = y^T D y = -1 \cdot u^2 - 1 \cdot v^2 + 2w^2 = -u^2 - v^2 + 2w^2$

5.5 a) $A = \begin{pmatrix} 0.4 & 0.15 \\ 0.6 & 0.85 \end{pmatrix}$ Markov ch. matrix

regular since $A > 0$

$\lambda = 1$: $\begin{pmatrix} -0.6 & 0.15 \\ 0.6 & -0.15 \end{pmatrix} = \begin{pmatrix} \textcircled{-0.6} & 0.15 \\ 0.6 & -0.15 \end{pmatrix}$ $\quad -0.6x + 0.15y = 0$

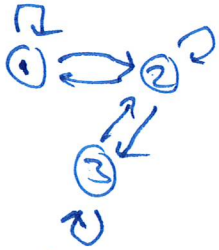
$x = \frac{0.15y}{0.6} = \frac{1}{4} y$

E_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \cdot y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$ $\quad y$ free

$1/4 + 1 = 5/4 \Rightarrow v = \frac{1}{5/4} \cdot \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = 4/5 \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$ eq. state

f) $A = \begin{pmatrix} 0.2 & 0.4 & 0 \\ 0.8 & 0.4 & 0.7 \\ 0 & 0.2 & 0.3 \end{pmatrix}$ Markov ch. matrix

regular? yes since $A^2 > 0$



$\lambda = 1$: $\begin{pmatrix} \textcircled{-0.8} & 0.4 & 0 \\ 0.8 & -0.6 & 0.7 \\ 0 & 0.2 & -0.7 \end{pmatrix} \begin{matrix} \\ +1 \\ \\ \end{matrix} \to \begin{pmatrix} \textcircled{-0.8} & 0.4 & 0 \\ 0 & \textcircled{-0.2} & 0.7 \\ 0 & 0.2 & -0.7 \end{pmatrix}$

$-0.8x + 0.4y = 0$

$-0.2y + 0.7z = 0$

z free

$y = \frac{0.7z}{0.2} = 3.5z$

$x = \frac{0.4(3.5z)}{0.8} = 1.75z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.75z \\ 3.5z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1.75 \\ 3.5 \\ 1 \end{pmatrix}$ eq. st.

$v = \frac{1}{6.25} \begin{pmatrix} 1.75 \\ 3.5 \\ 1 \end{pmatrix} = \frac{4}{25} \begin{pmatrix} 7/4 \\ 7/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/25 \\ 14/25 \\ 4/25 \end{pmatrix}$

6.2 b) $f = 2xz - 2yw - x^2 - y^2 - z^2 - w^2$

$= x^T A x$

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

$D_1 = -1$, $D_2 = 1$, $D_3 = 0$, $D_4 = 0$

$f'(x) = 2Ax = 0$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rk} A = 2$ $\Rightarrow$ neg. RRC smdt.

$(0,0,0,0)$ is max pt. $\Leftarrow$ all stat pts of A are max $\Leftarrow$ f concave

$f_{max} = f(0,0,0,0) = 0$

c) $f = 2xy + 2xz + 2yw + 2zw$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$D_1 = 0$, $D_2 = -1$ indefin.

no max/min

all stat pts are saddle pts by the second derivative test

d) $f = x^2 + y^2 + z^2 + w^2 + xy - yz + zw$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

pos. defin.

$$H(4) = 2A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$D_1 = 2$, $D_2 = 3$, $D_3 = -1 \cdot 2 + 2 \cdot 3 = 4$, $D_4 = -1 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} + 2 \cdot 4 = 8 - 1 \cdot 3 = 5$

pos. defin.

$f_{min} = f(0,0,0,0) = 0$

f convex

6.3 b) $f = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$= x^T A x$

$$f'(x) = 2A \cdot x = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \cdot x = 0$$

$= 2A$

$|2A| = 5 \neq 0 \Rightarrow x = 0$ stat. pt

0 is local min since f is convex and global min

d) $f = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

$f'_x = -4x^3 - 4x + 6z = 0$

$f'_y = -6y = 0$

$f'_z = 6x - 12z = 0$

$$H(f) = \begin{pmatrix} -12x^2 - 4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

Stat pts:

$y = 0 \quad \frac{6x}{6} = \frac{12z}{6} \quad x = 2z = 0$

$-4(2z)^3 - 4(2z) + 6z = 0$

$-32z^3 - 2z = 0$

$-2z(16z^2 + 1) = 0$

$z = 0$ or ~~$16z^2 + 1 = 0$~~

Stat pts: $(0,0,0)$

$$H(f)(0,0,0) = \begin{pmatrix} -4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$D_1 = -4$

$D_2 = 24$

$D_3 = -6(48 - 36) = -72$

neg. delta $\Rightarrow (0,0,0)$ local max

$$H(\lambda) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -2 \end{pmatrix}$$

$D_1 = -12x^2 - 4 < 0$ for all x,y,z

$D_2 = -6 D_1 \geq 0$ — " —

$D_3 = -6\left(-12(-12x^2-4) - 36\right)$

$= -6(144x^2 + 48 - 36)$

$= -6(144x^2 + 12) <$ — " —

neg. def. for all x,y,z

$\Downarrow$

f concave

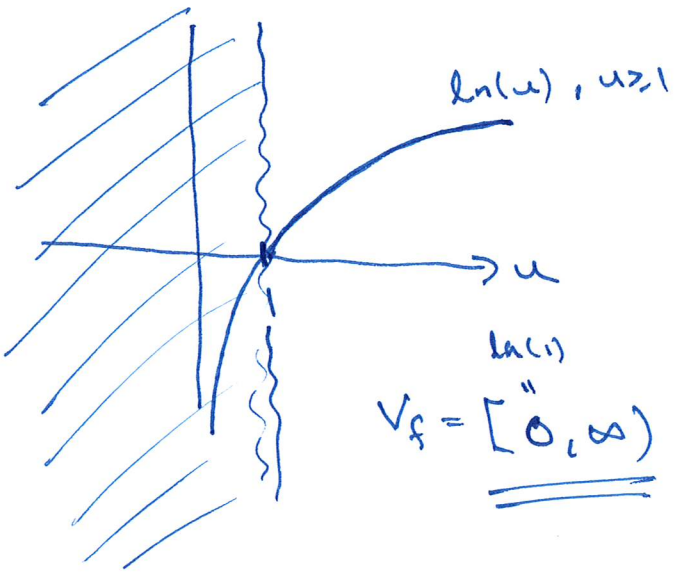
$\Downarrow$

$(0,0,0)$ global max pt

6.5 a) $f = \ln(1+2x^2+2xy+3y^2-2xz+z^2)$

$= \ln(u)$, $u = 1+2x^2+2xy+3y^2-2xz+z^2$

$= 1 + \underline{x}^T A \underline{x}$



$V_f = [0, \infty)$

$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ $D_1 = 2$, $D_2 = 5$, $D_3 = -1(3)+1\cdot 5 = 2$

pos. def.

$H(f) = 2A$ pos. def. $\Rightarrow$ f convex

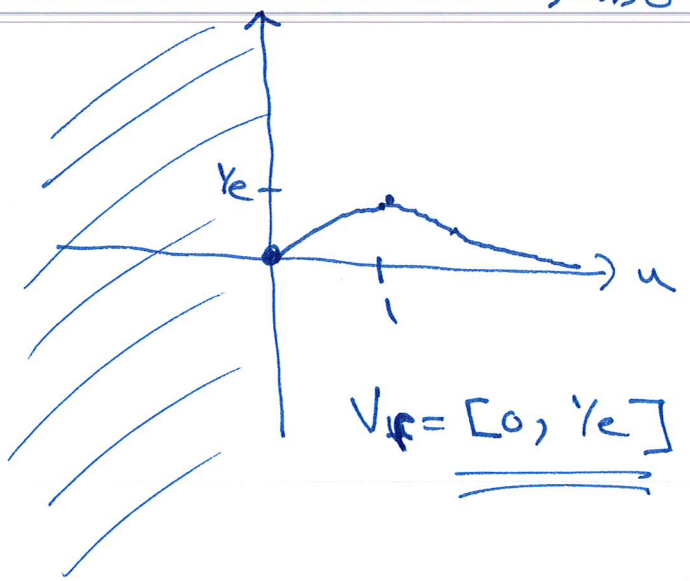
min: $u_{min} = u(0,0,0) = 1$

$u \geq 1$ $V_u = [1, \to)$

b) $f = (x^2+y^2+z^2)e^{-(x^2+y^2+z^2)} = ue^{-u}$, $x^2+y^2+z^2 = u$

$f(u) = ue^{-u}$, $u \geq 0$

$u = x^2+y^2+z^2$ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$V_f = [0, 1/e]$

u convex pos. def.

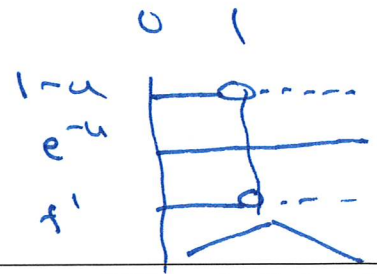
min: $u_{min} = u(0,0,0) = 0$

$u \geq 0$

$f(0) = 0\cdot e^{-0} = 0$ $f(1) = 1\cdot e^{-1} = 1/e$

$f'(u) = (ue^{-u})' = 1\cdot e^{-u} + u\cdot e^{-u}\cdot(-1)$

$= e^{-u} - u\cdot e^{-u} = (1-u)e^{-u}$



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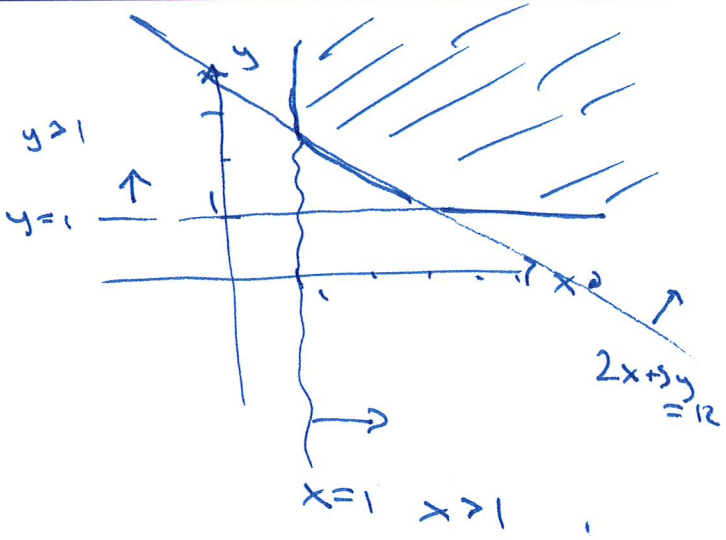
7.1 c) $D: x, y \geq 1$

$2x+3y \geq 12$

D is closed

not bounded

$\Rightarrow$ not compact



$2x+3y=12$ line thr. (6,0) and (0,4)

7.2 b) $3x^2+2y^2+2z^2=12$ compact set $(x^2 \leq 4,\ y^2, z^2 \leq 6)$

$L = x^2+y^2+z^2-\lambda(3x^2+2y^2+2z^2-12)$

$L'_x = 2x-\lambda\cdot 6x = 0$ $\quad 2x(1-3\lambda)=0$ $\quad x=0$ or $\lambda=1/3$

$L'_y = 2y-\lambda\cdot 4y = 0$ $\quad 2y(1-2\lambda)=0$ $\quad y=0$ " $\lambda=1/2$

$L'_z = 2z-\lambda\cdot 4z = 0$ $\quad 2z(1-2\lambda)=0$ $\quad z=0$ " $\lambda=1/2$

$3x^2+2y^2+2z^2=12$

Cases:

(A) $\lambda=1/3$: $y=z=0$ $\quad 3x^2=12$ $x^2=4$ $\quad (\pm 2,0,0;\ 1/3)$ $f=4$

(B) $\lambda=1/2$: $x=0$, $2y^2+2z^2=12$ $y^2+z^2=6$ $\quad f=6$ $\quad (0,y,z;\ 1/2)$ $y^2+z^2=6$

(C) $\lambda \neq 1/2, 1/3$: $x=y=z=0$ $\quad 0 \neq 12$ no cand pts

$f_{max}=6$ at $(0,y,z)$ with $y^2+z^2=6$

$f_{min}=4$ at $(\pm 2,0,0)$

NDCQ: $\operatorname{rk}(4x\ \ 4y\ \ 4z)=1$

NDCQ fails: $\operatorname{rk}=0$ $\quad x=0, y=0, z=0$ not adm. $0 \neq 12$

3 b) max/min $f = xw - yz$ s.t. $\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$

$$L = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$

(1) $L'_x = w - \lambda_1 \cdot 2x = 0$
(2) $L'_y = -z - \lambda_1 \cdot 8y = 0$
(3) $L'_z = -y - \lambda_2 \cdot 8z = 0$
(4) $L'_w = x - \lambda_2 \cdot 18w = 0$

$x^2+4y^2 = 4$
$4z^2+9w^2 = 36$

(1) and (4):

$w = 2\lambda_1 x$

$x - 18\lambda_2(2\lambda_1 x) = 0$

$x(1-36\lambda_1\lambda_2) = 0$

$x=0$, $w=0$ or $\lambda_1\lambda_2 = 1/36$

(2) and (3):

$z = -8\lambda_1 y \Rightarrow \lambda_1 = \frac{z}{-8y}$

$-y - 8\lambda_2(-8\lambda_1 y) = 0$

$-y(1-64\lambda_1\lambda_2) = 0$

$y=0$, $z=0$ or $\lambda_1\lambda_2 = 1/64$

Cases:

A) $x=0, y=0$, $w=0, z=0$ — $0+0=4$ not possible

B) $x=0$, $\lambda_1\lambda_2 = 1/64$, $w=0$: $4y^2=4$, $y^2=1$, $y=\pm 1$; $4z^2=36$, $z^2=9$, $z=\pm 3$

C) $\lambda_1\lambda_2 = 1/36$, $y=0$

$(0,1,3,0; -3/8, -1/24)$ $f=-3$
$(0,1,-3,0; 3/8, 1/24)$ $f=3$
$(0,-1,3,0; 3/8, 1/24)$ $f=3$
$(0,-1,-3,0; -3/8, -1/24)$ $f=-3$

Use SOC to test cand pts with $f=3$:

$h = xw - yz - \frac{3}{8}(x^2+4y^2-4) - \frac{1}{24}(4z^2+9w^2-36)$

$= \frac{12}{8} + \frac{36}{24} + \underline{x}^T A \underline{x}$

$$A = \begin{pmatrix} -3/8 & 0 & 0 & 1/2 \\ 0 & -3/2 & -1/2 & 0 \\ 0 & -1/2 & -1/6 & 0 \\ 1/2 & 0 & 0 & -3/8 \end{pmatrix}$$

$D_1 = -3/8 < 0$
$D_2 = 9/16 > 0$
$D_3 = -3/8(3/12 - 1/4) = 0$
$D_4 = 0$

$\Delta_1 = -3/8, -3/2, -1/6, -3/8 < 0$

$\Delta_2 = 9/16, 0, 3/48, 3/24, 2/16, 9/64 - 1/4 = -7/64$

indefin. $\Rightarrow$ SOC does not work for these pts

Try case C:

C) $\lambda_1\lambda_2 = 1/36$, $y = 0$

$z = 0$ $x^2 = 4$ $x = \pm 2$

$9w^2 = 36$ $w^2 = 4$ $w = \pm 2$

$\lambda_1 = \frac{w}{2x}$ from (1)

$(2,0,0,2; 1/2, 1/18)$ $f = 4$

$(2,0,0,-2; -1/2, -1/18)$ $f = -4$

$(-2,0,0,2; -1/2, -1/18)$ $f = -4$

$(-2,0,0,-2; 1/2, 1/18)$ $f = 4$

SOC for pts with $f = 4$:

$h = h(x,y,z,w; 1/2, 1/18) = xw - yz - \frac{1}{2}(x^2+4y^2-4) - \frac{1}{18}(4z^2+9w^2-36)$

$= 2 + 2 + \underline{x}^T A \underline{x}$ with $A = \begin{pmatrix} -1/2 & 0 & 0 & 1/2 \\ 0 & -2 & -1/2 & 0 \\ 0 & -1/2 & -2/9 & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix}$

rk $A = 3$, $\Rightarrow$ $D_4 = -1/2 < 0$

RRC: A neg. semidefin. $D_2 = 1 > 0$

$\Rightarrow$ h concave $D_3 = -\frac{1}{2}(4/9 - 1/4)$

$\Rightarrow f_{max} = 4$ $= -\frac{1}{2}\left(\frac{16-9}{36}\right) < 0$

$D_4 = 0$ since $R_4 = -R_1$