NORWEGIAN BUSINESS SCHOOL

Evaluation guidelines:	**GRA 60353**	**Mathematics**	
Examination date:	13.12.2012	09:00 – 12:00	Total no. of pages: 4
Permitted examination support material:	A bilingual dictionary and BI-approved calculator TEXAS INSTRUMENTS BA II Plus		
Answer sheets:	Squares		
	Counts 80% of GRA 6035		The subquestions are weighted equally
			Responsible department: Economics

Question 1.

(a) We compute the determinant of A using cofactor expansion along the first column, and find that

$$\det(A) = \begin{vmatrix} t & 1 & 1 \\ t & 2 & 1 \\ 4 & t & 2 \end{vmatrix} = t(4-t) - t(2-t) + 4 \cdot (-1) = \mathbf{2t - 4}$$

Since $\det(A) \neq 0$ for $t \neq 2$, and the minor $\left|\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right| = -1$ of order two is non-zero, we have that

$$\operatorname{rk}(A) = \begin{cases} \mathbf{3}, & t \neq 2 \\ \mathbf{2}, & t = 2 \end{cases}$$

(b) When $t = -2$, the characteristic equation of A is given by

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 1 \\ -2 & 2-\lambda & 1 \\ 4 & -2 & 2-\lambda \end{vmatrix} = 0$$

Cofactor expansion along the first column gives

$$(-2-\lambda)((2-\lambda)^2 + 2) - (-2)(2 - \lambda + 2) + 4(1 - (2-\lambda)) = 0$$

and we find that this reduces to

$$(-2-\lambda)(2-\lambda)^2 + 2(-2-\lambda) + 2(4-\lambda) + 4(\lambda - 1) = (-2-\lambda)(2-\lambda)^2 = 0$$

The eigenvalues are therefore $\lambda = \mathbf{-2}$ and $\lambda = \mathbf{2}$, where the last eigenvalue has multiplicity two. When $\lambda = 2$, the eigenvectors are given by $(A - 2I)\mathbf{x} = \mathbf{0}$, and the matrix

$$A - 2I = \begin{pmatrix} -4 & 1 & 1 \\ -2 & 0 & 1 \\ 4 & -2 & 0 \end{pmatrix}$$

has rank two since $A - 2I$ has a non-zero minor $\left|\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right| = 1$ of order two — it cannot have rank three since $\lambda = 2$ is an eigenvalue. Therefore, the linear system has just one free variable while $\lambda = 2$ is an eigenvalue of multiplicity two. So A is **not diagonalizable** when $t = -2$.

Question 2.

(a) We compute the partial derivatives and the Hessian matrix of f:

$$\begin{pmatrix} f'_x \\ f'_y \\ f'_z \end{pmatrix} = \begin{pmatrix} -4x^3 - 2hx + 6z \\ -6y \\ 6x - 12z \end{pmatrix}, \quad f'' = \begin{pmatrix} -12x^2 - 2h & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

We see that the leading principal minors are given by $D_1 = -12x^2 - 2h$, $D_2 = -6D_1$ and $D_3 = -6(144x^2 + 24h - 36)$. Hence $D_1 \leq 0$ for all (x, y, z) if and only if $h \geq 0$, and if this is the case then $D_2 = -6D_1 \geq 0$. Moreover, $D_3 \leq 0$ for all (x, y, z) if and only if $h \geq 3/2$. This means that $D_1 \leq 0,\ D_2 \geq 0,\ D_3 \leq 0$ if and only if $h \geq 3/2$, and the equalities are strict if $h > 3/2$. If $h = 3/2$, then $D_3 = 0$, and we compute the remaining principal minors. We find that $\Delta_1 = -6, -12 \leq 0$ and that $\Delta_2 = 144x^2, 72 \geq 0$. We conclude that f is concave if and only if $h \geq 3/2$, and $H = \mathbf{3/2}$.

(b) We compute the stationary points, which are given by the equations

$$-4x^3 - 2hx + 6z = 0, \quad -6y = 0, \quad 6x - 12z = 0$$

The last two equations give $y = 0$ and $z = x/2$, and the first equations becomes

$$-4x^3 - 2hx + 3x = x(-4x^2 + 3 - 2h) = 0 \quad \Leftrightarrow \quad x = 0$$

since $x^2 = (3 - 2h)/4$ has no solutions when $h > 3/2$ and the solution $x = 0$ when $h = 3/2$. The stationary points are therefore given by $(x^*(h), y^*(h), z^*(h)) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ when $h \geq 3/2$, and this is the global maximum since f is concave.

(c) Let $h \geq 3/2$. By the Envelope Theorem, we have that

$$\frac{d}{dh} f^*(h) = \frac{\partial f}{\partial h}\Big|_{(x,y,z)=(0,0,0)} = (-x^2 + 2h)\big|_{(x,y,z)=(0,0,0)} = 2h \geq 3$$

Since the derivative is positive, the maximal value will **increase** when h increases. We could also compute $f^*(h) = f(0, 0, 0) = 12 + h^2$ explicitly for $h \geq 3/2$, and use this to see that $f^*(h)$ increases when h increases.

Question 3.

(a) The homogeneous equation $y'' - 5y' + 6y = 0$ has characteristic equation $r^2 - 5r + 6 = 0$, and therefore roots $r = 2, 3$. Hence the homogeneous solution is $y_h(t) = C_1 e^{2t} + C_2 e^{3t}$. To find a particular solution of $y'' - 5y' + 6y = 10e^{-t}$, we try $y = Ae^{-t}$. This gives $y' = -Ae^{-t}$ and $y'' = Ae^{-t}$, and substitution in the equation gives $(A + 5A + 6A)e^{-t} = 10e^{-t}$, or $12A = 10$. Hence $A = 5/6$ is a solution, and $y_p(t) = \frac{5}{6}e^{-t}$ is a particular solution. This gives general solution

$$y(t) = \mathbf{C_1 e^{2t} + C_2 e^{3t} + \frac{5}{6} e^{-t}}$$

(b) The differential equation $4te^{2t}y - (1 - 2t)e^{2t}y' = 0$ is exact if and only if there is a function $h(t, y)$ such that

$$\frac{\partial h}{\partial t} = 4te^{2t}y, \quad \frac{\partial h}{\partial y} = -(1 - 2t)e^{2t}$$

We see that $h(t, y) = -(1 - 2t)e^{2t}y$ is a solution to the last equation, and differentiation shows that it is a solution to the first equation as well. Therefore the solution of the exact differential equation is given by

$$h(t, y) = -(1 - 2t)e^{2t}y = C \quad \Rightarrow \quad y = \frac{\mathbf{Ce^{-2t}}}{\mathbf{2t - 1}} \quad \text{(when } t > 1/2\text{)}$$

Question 4.

(a) The homogeneous equation $p_{t+2} - 2p_{t+1} + p_t = 0$ has characteristic equation $r^2 - 2r + 1 = 0$, with double root $r = 1$. Therefore, the homogeneous solution is $p_t^h = (C_1 + C_2 t)1^t = C_1 + C_2 t$. To find a particular solution, we first try $p_t = A$, which gives $0 = -15$ and there is no solution for A. We then try $p_t = At$, and get $A(t+2) - 2A(t+1) + At = -15$, or $0 = -15$, and there is again no solution for A. We try $p_t = At^2$, and get $A(t+2)^2 - 2A(t+1)^2 + At^2 = -15$, or $2A = -15$. The solution is $A = -7.5$, and we get $p_t = C_1 + C_2 t - 7.5t^2$. The initial conditions give $C_1 = 695$ and $695 + C_2 - 7.5 = 743$, or $C_2 = 55.5$. The solution of the difference equation is therefore $p_t = \mathbf{695 + 55.5t - 7.5t^2}$. Alternatively, the difference equation can be solved using the difference $d_t = p_{t+1} - p_t$. With this method, we first find d_t (see below), and then solve the first order difference equation $p_{t+1} - p_t = d_t$ when d_t is known.

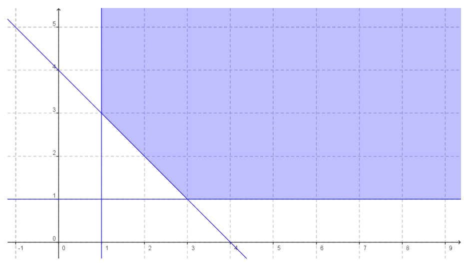
(b) Let $d_t = p_{t+1} - p_t$ be the increase in the housing prices p_t from year t to $t+1$. Then we can rewrite the difference equation as

$$d_{t+1} - d_t = (p_{t+2} - p_{t+1}) - (p_{t+1} - p_t) = p_{t+2} - 2p_{t+1} + p_t = -15$$

This result can also be obtained from the expression for p_t found above. We can use this to determine d_t, since we have a first order difference equation $d_{t+1} - d_t = -15$, with initial condition $d_0 = p_1 - p_0 = 48$. We get homogeneous solution $d_t^h = C1^t = C$. To find a particular solution, we first try $d_t = A$. Since this gives $0 = -15$, we try $d_t = At$, and get $A = -15$. So the general solution is $d_t = C - 15t$, and the initial condition $d_0 = 48$ gives $C = 48$. Alternatively, we can see directly that the solution for d_t is given by $d_t = 48 - 15t$, since d_t is an arithmetic sequence. We conclude that $d_t > 0$ for $t = 0, 1, 2, 3$ and that $d_t < 0$ for $t \geq 4$. This means that the housing prices increases in the first 4 years (from $t = 0$ to $t = 4$) and decreases after that (from $t = 4$).

Question 5.

For the sketch, see the figure below. Since $\ln(ab) = \ln(a) + \ln(b)$, we can rewrite the function as



$f(x, y) = 2\ln x + \ln y - x - y$ and the constraints as $-x - y \leq -4, -x \leq -1, -y \leq -1$. We write the Lagrangian for this problem as

$$\begin{aligned} \mathcal{L} &= 2\ln x + \ln y - x - y - \lambda(-x - y) - \nu_1(-x) - \nu_2(-y) \\ &= 2\ln x + \ln y - x - y + \lambda(x + y) + \nu_1 x + \nu_2 y \end{aligned}$$

The Kuhn-Tucker conditions for this problem are the first order conditions

$$\begin{aligned} \mathcal{L}'_x &= \frac{2}{x} - 1 + \lambda + \nu_1 = 0 \\ \mathcal{L}'_y &= \frac{1}{y} - 1 + \lambda + \nu_2 = 0 \end{aligned}$$

the constraints $x + y \geq 4$ and $x, y \geq 1$, and the complementary slackness conditions $\lambda, \nu_1, \nu_2 \geq 0$ and

$$\lambda(x + y - 4) = 0, \quad \nu_1(x - 1) = 0, \quad \nu_2(y - 1) = 0$$

Let us find all solutions of the Kuhn-Tucker conditions: If $x = 1$, then $1 + \lambda + \nu_1 = 0$ by the first FOC and this is not possible (since $\lambda, \nu_1 \geq 0$). So we must have $x > 1$ and $\nu_1 = 0$. If $y = 1$, then $\lambda + \nu_2 = 0$ by the second FOC, and this implies that $\lambda = \nu_2 = 0$ (since $\lambda, \nu_2 \geq 0$). Then the first

FOC implies that $x = 2$, and this is not possible since $x + y \geq 4$. Hence we must also have $y > 1$ and $\nu_2 = 0$. Using the FOC's, we get

$$\lambda = 1 - \frac{2}{x} = 1 - \frac{1}{y}$$

which gives $2/x = 1/y$ or $x = 2y$ and $\lambda = 1 - 1/y > 0$ since $y > 1$. This implies that $x + y = 4$, which gives $3y = 4$ or $y = 4/3$, $x = 8/3$ and $\lambda = 1/4$. We conclude that there is exactly one solution of the Kuhn-Tucker conditions:

$$(x, y; \lambda, \nu_1, \nu_2) = (8/3, 4/3; 1/4, 0, 0)$$

The Lagrangian $\mathcal{L} = \mathcal{L}(x, y; 1/4, 0, 0) = 2 \ln x + \ln y - x - y + (x + y)/4$ has Hessian

$$\mathcal{L}'' = \begin{pmatrix} -\frac{2}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$$

so $\mathcal{L}$ is a concave function, since $D_1 = -2/x^2 < 0$ and $D_2 = 2/(x^2 y^2) > 0$ ($\mathcal{L}$ is only defined for $x, y \neq 0$). Therefore $(x, y) = (\mathbf{8/3}, \mathbf{4/3})$ is the maximum point.