BI NORWEGIAN BUSINESS SCHOOL

Solutions:	**GRA 60353 Mathematics**	
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Answer sheets:	Squares	
	Counts 80% of GRA 6035	The subquestions have equal weight
Re-take exam		Responsible department: Economics

Question 1.

(a) Gaussian elimination of A and $A+I$ to echelon form gives

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A+I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, we have $\operatorname{rk} A = 3$ and $\operatorname{rk}(A+I) = 1$.

(b) The characteristic equation for A is given by

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2-1) - 1(-\lambda-1) + 1(1+\lambda) = (\lambda+1)(-\lambda(\lambda-1)+2) = 0$$

Therefore, the eigenvalues of A are given by $\lambda = -1$ and $-\lambda^2+\lambda+2 = 0$, and the last equation gives

$$\lambda = \frac{-1 \pm \sqrt{9}}{-2} = \frac{-1 \pm 3}{-2} = 2, -1$$

The conclusion is that the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 2$.
Alternative: It is possible to see that $\lambda = -1$ is an eigenvalue of multiplicity 2 from the fact that $\operatorname{rk}(A+I) = 1$, and since the sum of the eigenvalues is $(-1)+(-1)+\lambda = \operatorname{tr}(A) = 0$ we can conclude that the last eigenvalue is $\lambda = 2$. We could also have used that the product of the eigenvalues is $(-1)(-1)\lambda = \det A = 2$ to find that $\lambda = 2$.

(c) The matrix A is symmetric, therefore it is diagonalizable.
Alternative: Since $\lambda = -1$ has multiplicity two while $\lambda = 2$ has multiplicity one, we check that the number of degrees of freedom of the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda = -1$: It has two degrees of freedom, since $\operatorname{rk}(A+I) = 1$ and $n - \operatorname{rk}(A+I) = 3 - 1 = 2$. Therefore there are enough eigenvalues and enough eigenvectors, and A is diagonalizable.

Question 2.

(a) The partial derivatives of $f(x,y,z,w) = x^3 + 3xy^2 - 3x - 2z^3 + 6zw^2 - 3w$ are given by

$$f'_x = 3x^2 + 3y^2 - 3, \quad f'_y = 6xy, \quad f'_z = -6z^2 + 6w^2, \quad f'_w = 12zw - 3$$

and its Hessian matrix is given by

$$H(f)(x,y,z,w) = \begin{pmatrix} 6x & 6y & 0 & 0 \\ 6y & 6x & 0 & 0 \\ 0 & 0 & -12z & 12w \\ 0 & 0 & 12w & 12z \end{pmatrix}$$

(b) The stationary points of f are given by

$$f'_x = 3x^2 + 3y^2 - 3 = 0, \quad f'_y = 6xy = 0, \quad f'_z = -6z^2 + 6w^2 = 0, \quad f'_w = 12zw - 3 = 0$$

From the first two equations, we get $x^2 + y^2 = 1$ and $xy = 0$, which gives $(x,y) = (\pm 1, 0)$ or $(x,y) = (0, \pm 1)$. From the last two equations, we get $z^2 = w^2$ and $zw = 1/4$, which gives $z = \pm w$, and since $zw = 1/4 > 0$, it must be $z = w$. Finally, $zw = z^2 = 1/4$ gives $z = w = \pm 1/2$. The stationary points are therefore the eight points

$$(x,y,z,w) = (\pm 1, 0, 1/2, 1/2), (\pm 1, 0, -1/2, -1/2), (0, \pm 1, 1/2, 1/2), (0, \pm 1, -1/2, -1/2)$$

When $(x,y) = (0, \pm 1)$, the Hessian matrix is given by

$$H(f)(0, \pm 1, z, w) = \begin{pmatrix} 0 & \pm 6 & 0 & 0 \\ \pm 6 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

and has $D_2 = -36 < 0$. The four stationary points with $(x,y) = (0, \pm 1)$ are therefore saddle points. When $(x,y) = (\pm 1, 0)$ and $(z,w) = (1/2, 1/2)$ or $(-1/2, -1/2)$, we get

$$D_4 = \begin{vmatrix} 6x & 0 & 0 & 0 \\ 0 & 6x & 0 & 0 \\ 0 & 0 & -12z & 12w \\ 0 & 0 & 12w & 12z \end{vmatrix} = 36x^2(-144z^2 - 144w^2) < 0$$

since $x^2 = 1$ and $z^2 = w^2 = 1/4$. The four stationary points with $(x,y) = (\pm 1, 0)$ are therefore also saddle points. We conclude that all stationary points are saddle points.

(c) The function f is not concave. If it were concave, then $D_1 = f''_{xx} \geq 0$ for all (x,y,z,w) but this is not the case since $f''_{xx} = 6x$. Another argument is that if f were concave, then all stationary points would be (local and global) maximum points, but this is not the case.

Question 3.

(a) The difference equation $y_{t+1} - 2y_t = 3t$ is first order linear, and it has solution $y_t = y_t^h + y_t^p$. The homogeneous equation $y_{t+1} - 2y_t = 0$ has solution $y_t^h = C \cdot 2^t$. To find a particular solution y_t^p, we consider the right hand side $f_t = 3t$ and the shifted expressions $f_{t+1} = 3t + 3$. We guess that there is a solution of the form $y_t = At + B$. Inserting this guess in the difference equation, we obtain

$$(At + B + A) - 2(At + B) = 3t$$

or $(-A)t + (A - B) = 3t$. We see that $A = B = -3$ is a solution, so $y_t^p = -3t - 3$ and the general solution is

$$y_t = y_t^h + y_t^p = C \cdot 2^t - 3t - 3$$

(b) The differential equation $y'' - 12y' + 20y = 2te^t$ is second order linear, and it has solution $y = y_h + y_p$. The homogeneous equation $y'' - 12y' + 20y = 0$ has characteristic equation given by $r^2 - 12r + 20 = 0$, and has solutions $r = 2, 10$, so the homogeneous solution is $y_h = C_1 e^{2t} + C_2 e^{10t}$. To find a particular solution, we consider the right hand side $f(t) = 2te^t$

and its derivatives $f' = (2t+2)e^t$ and $f'' = (2t+4)e^t$. We guess that there is a solution of the form $y = (At+B)e^t$. Inserting this guess in the differential equation, we obtain

$$(At+B+2A)e^t - 12(At+B+A)e^t + 20(At+B)e^t = 2te^t$$

or $(9At+9B-10A)e^t = 2te^t$. We see that there is a solution with $9A = 2$ and $9B-10A = 0$, or $A = 2/9$ and $B = 20/81$. This means that $y_p = (2/9t + 20/81)e^t$ is a particular solution, and that the general solution is

$$y = y_h + y_p = C_1e^{2t} + C_2e^{10t} + \frac{18t+20}{81}\,e^t$$

(c) The differential equation $y' + \ln(t)\,y = \ln(t)$ is first order linear, and it is in standard form $y' + a(t)y = b(t)$ with $a(t) = b(t) = \ln t$. It has integrating factor $u = e^{\int a(t)\,\mathrm{d}t}$, and

$$\int \ln(t)\,\mathrm{d}t = t\ln(t) - t + \mathcal{C}$$

We therefore multiplity the differential equation with $u = e^{t\ln t - t}$, and get

$$(y\,e^{t\ln t-t})' = \ln(t)\,e^{t\ln t-t} \quad\Rightarrow\quad y\,e^{t\ln t-t} = \int \ln(t)\,e^{t\ln t-t}\,\mathrm{d}t = e^{t\ln t-t} + \mathcal{C}$$

We have solved the integral using the substitution $v = t\ln t - t$ which gives $\mathrm{d}v = \ln t\,\mathrm{d}t$. This implies that the solution of the differential equation is

$$y = \frac{e^{t\ln t-t} + \mathcal{C}}{e^{t\ln t-t}} = 1 + C\,e^{-t\ln t+t}$$

for $t > 0$.

Question 4.

(a) The Kuhn-Tucker problem is already in standard form, so we form the Lagrangian

$$\mathcal{L} = x^3 + y^3 + z^3 - 3xyz - \lambda(x^3 + y^3 + z^3)$$

The first order conditions (FOC) are

$$\begin{aligned}
\mathcal{L}'_x &= 3x^2 - 3yz - 3\lambda x^2 = 0\\
\mathcal{L}'_y &= 3y^2 - 3xz - 3\lambda y^2 = 0\\
\mathcal{L}'_z &= 3z^2 - 3xy - 3\lambda z^2 = 0
\end{aligned}$$

the constraint (C) is given by $x^3 + y^3 + z^3 \le 8$, and the complementary slackness conditions (CSC) are given by

$$\lambda \ge 0 \quad \text{and } \lambda(x^3 + y^3 + z^3 - 8) = 0$$

When $\lambda = 1$, the FOC's are given by $yz = xz = xy = 0$, which means that at least two of the variables are zero, and the CSC means that $x^3 + y^3 + z^3 = 8$. If $y = z = 0$, then $x^3 = 8$ and therefore $x = 2$, and the two other cases are similar. We find exactly three solutions to the KT conditions when $\lambda = 1$,

$$(x, y, z) = (2, 0, 0),\ (0, 2, 0),\ (0, 0, 2)$$

and $f = 8$ at all three points.

(b) An admissible point is one where the constraint $x^3 + y^3 + z^3 \le 8$ is satisfied. The NDCQ is given by $\operatorname{rk} J(x, y, z) = 1$ when $x^3+y^3+z^3 = 8$ (constraint is binding), and there is no NDCQ condition when $x^3+y^3+z^3 < 8$ (constraint is non-binding). The matrix $J(x, y, z)$ is given by the partial derivatives of the constraint:

$$J(x, y, z) = \begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \end{pmatrix}$$

The only possibility for NDCQ to fail is if $\operatorname{rk} J = 0$ and the constrating is binding. This will not happen, since $\operatorname{rk} J = 0$ only if $3x^2 = 3y^2 = 3z^2 = 0$, or if $(x, y, z) = (0, 0, 0)$, and the constraint is not binding at this point. Therefore, NDCQ is satisfied at all admissible points.

(c) The set of admissible points (points that satisfy $x^3 + y^3 + z^3 \leq 8$) is not bounded, since the points $(x, y, z) = (a, 0, 0)$ satisfies the constraint as long as $a^3 \leq 8$, or $a < 2$, and this includes all negative values of a.

(d) The Kuhn-Tucker problem does not have a solution. We prove this by finding admissible points where the function value is arbitary large (goes towards infinity):
The point $(x, y, z) = (2, b, -b)$ satisfy the constraint $x^3 + y^3 + z^3 \leq 8$ for any value of b, since $2^3 + b^3 + (-b)^3 = 2^3 = 8$. In other words, the point $(x, y, z) = (2, b, -b)$ is admissible for any value of b. The function value is given by

$$f(2, b, -b) = 2^3 + b^3 + (-b)^3 - 3 \cdot 2b(-b) = 8 + 6b^2$$

Note that $f(2, b, -b) = 8 + 6b^2 \to \infty$ as $b \to \infty$, and that $(2, b, -b)$ is admissible also when b is very large. This proves that the Kuhn-Tucker problem has no solution.