Exam	Final exam in GRA 6035 Mathematics
Date	December 12th, 2014 at 0900 - 1200

All subquestions have the same weight and give maximal score 6p each. Answers to the first 12 subquestions give a maximal score of 72p (100%). Question 5 can be skipped, but gives 6p extra credit if answered correctly.

Question 1.

We consider the function given by $f(x, y, z) = x^4 + y^2 - xz + z^4$.

(a) **(6p)** Compute the partial derivatives and the Hessian matrix of f.
(b) **(6p)** Find all stationary points of f, and classify them as local max, local min or saddle points.
(c) **(6p)** Is f convex?

Question 2.

We consider the matrix A given by

$$A = \begin{pmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{pmatrix}$$

(a) **(6p)** Compute the determinant of A, and the rank of $A - \lambda I$ when $\lambda = t - 1$.
(b) **(6p)** Show that A is diagonalizable when $t = 8$, and find all its eigenvalues in this case.

A car rental firm has three locations and 120 cars. We assume that all cars are returned after one week, and that any rented car is 8 times as likely to be returned to the pick-up location as any of the other locations.

(c) **(6p)** Find the transition matrix of the resulting Markov chain. If one of the three locations is at an airport, and 50 of the cars are starting out at this location, how many cars will be at the airport location in the long run?

Question 3.

Solve the difference equation:

(a) **(6p)** $y_{t+1} - 3y_t = -5(2t + 1)$

Solve the differential equations:

(b) **(6p)** $t^3 y' = y^2$
(c) **(6p)** $(2yt - 1)y' = (t + 1)e^t - y^2$

Question 4.

We consider the following Kuhn-Tucker problem:

$$\max\ f(x, y, z, w) = x + 4y + 2z + 5w \text{ subject to } 2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 \le 21$$

(a) **(6p)** Write down the Kuhn-Tucker conditions for this problem, and find all points that satisfy these conditions. (You will find that the Lagrange multiplier $\lambda = 1/2$).
(b) **(6p)** Solve the Kuhn-Tucker problem and find the corresponding maximum value.
(c) **(6p)** Use (a) and (b) to estimate the maximum value in the Kuhn-Tucker problem

$$\max\ x + 3.8y + 2z + 5.4w \text{ subject to } 2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 \le 21$$

Question 5.

Extra credits (6p) We consider a Markov chain with an $n \times n$ transition matrix

$$T = \frac{1}{t+n-1}\begin{pmatrix} t & 1 & 1 & \dots & 1 \\ 1 & t & 1 & \dots & 1 \\ 1 & 1 & t & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & t \end{pmatrix}$$

for a positive integer $t > 1$. Show that the long run equilibrium state $\mathbf{x} = \lim_{n\to\infty} T^n \mathbf{x}_0$ of this Markov chain is

$$\mathbf{x} = \begin{pmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{pmatrix}$$

for all values of t.