Solutions	Final exam in GRA 6035 Mathematics
Date	January 6th, 2017 at 1200 - 1500

Question 1.

(a) The determinant of A is

$$\det(A) = \begin{vmatrix} -1 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1(1-1) = 0$$

This means that $\text{rk}(A) < 3$, and since at least one of the 2-minors are non-zero, for instance

$$\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$$

it follows that $\text{rk}(A) = 2$.

(b) The linear system $A\mathbf{x} = \mathbf{0}$ has one free variables since A has rank two, and we compute the solutions using Gaussian elimination:

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This means that z is free, $y = z$ and $x = y - z = 0$, and the solutions to the linear system can be written in the form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = z \cdot \mathbf{v}_1, \quad \text{with } \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

That is, the solutions are the vectors in $\text{span}(\mathbf{v}_1)$.

(c) To check if A is diagonalizable, we compute the eigenvalues of A, given by the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} -1-\lambda & 1 & -1 \\ -1 & -\lambda & 0 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

This gives the equation

$$(-1-\lambda)(-\lambda(1-\lambda)) + 1(1-\lambda-1) = 0$$

which gives, after multiplication, that

$$-\lambda(\lambda^2 - 1) - \lambda = -\lambda(\lambda^2 - 1 + 1) = -\lambda^3 = 0$$

The eigenvalues are therefore $\lambda = 0$ (with multiplicity three). The linear system that gives the eigenvectors for $\lambda = 0$, is given by

$$(A - 0 \cdot I)\,\mathbf{x} = \mathbf{0}$$

and since A has rank two, $A\mathbf{x} = \mathbf{0}$ has just one free variable. Since the multiplicity of $\lambda = 0$ is three, this means that A is not diagonalizable.

Question 2.

(a) The differential equation $y'' - 3y' - 10y = 20$ is second order linear, with solution $y = y_h + y_p$. The homogeneous equation $y'' - 3y' - 10y = 0$ has characteristic equation $r^2 - 3r - 10 = 0$, with solutions $r = 5$ and $r = -2$. Therefore, the general solution is

$$y_h = C_1 \cdot e^{5t} + C_2 \cdot e^{-2t}$$

Since the right hand side $f(t) = 20$ is a constant, we guess a constant solution $y_p = A$, which gives $y' = y'' = 0$ and $-10A = 20$, with solution $A = -2$. Therefore, the general solution is

$$y = y_h + y_p = C_1 \cdot e^{5t} + C_2 \cdot e^{-2t} - 2$$

(b) The differential equation $y' + \ln(t) = y\ln(t)$ is linear and separable since it can be written $y' - \ln(t)y = -\ln(t)$ or as $y' = (y-1)\ln t$. We solve it as a separable differential equation $y' = (y-1)\ln(t)$, which gives

$$\frac{1}{y-1}y' = \ln(t) \quad \Rightarrow \quad \int \frac{1}{y-1}\,\mathrm{d}y = \int \ln(t)\,\mathrm{d}t$$

Using integration by parts with $u' = 1$ and $v = \ln t$ to compute the last integral, we get

$$\ln|y-1| = t\ln t - t + C \quad \Rightarrow \quad |y-1| = e^{t\ln t - t}\cdot e^C$$

This implies that $y - 1 = Ke^{t\ln t - t}$ with $K = \pm e^c$, or that

$$y = Ke^{t\ln t - 1} + 1$$

If we would solve it as a linear differential equation $y' - \ln(t)y = -\ln(t)$ instead, the integrating factor u would be given by

$$\int -\ln(t)\mathrm{d}t = -t\ln t + t + C \quad \Rightarrow \quad u = e^{-t\ln t + t}$$

which gives

$$ye^{-t\ln t+t} = \int -\ln(t)e^{-t\ln t+t}\mathrm{d}t = \int e^u \mathrm{d}u = e^u + C = e^{-t\ln t+t} + C$$

with the substitution $u = -t\ln t + t$, $\mathrm{d}u = -\ln(t)\mathrm{d}t$. This would give

$$y = 1 + Ce^{t\ln t - t}$$

(c) We try to solve the differential equation $6t(y^2-t^2)^2 = 6y(y^2-t^2)^2 \cdot y'$ as an exact differential equation. We write it in the form $6t(y^2-t^2)^2 - 6y(y^2-t^2)^2\cdot y' = 0$, and try to find a function $h = h(y,t)$ such that

$$h'_t = 6t(y^2-t^2)^2, \quad h'_y = -6y(y^2-t^2)^2$$

From the first equation, it follows that $h = -(y^2-t^2)^3 + C(y)$, since the derivative $(-u^3)'_t = -3u^2 \cdot u'_t$ with $u = y^2 - t^2$ and $u'_t = -2t$. We check the second equation, and compute

$$h'_y = -3u^2 \cdot u'_y = -3(y^2-t^2)^2 \cdot 2y + C'(y)$$

Therefore $h = -(y^2-t^2)^3 + C(y)$ is a solution to both equations if $C'(y) = 0$, and the simplest solution to this is $C(y) = 0$. We therefore have that

$$h(y,t) = -(y^2-t^2)^3 = K \quad \Rightarrow \quad (y^2-t^2)^3 = -K$$

The initial condition $y(0) = 1$ gives $1 = -K$, or $K = -1$. Hence the solution is

$$y^2 - t^2 = \sqrt[3]{1} = 1 \quad \Rightarrow \quad y = \sqrt{t^2+1}$$

Notice that that we can exclude the possibility $y = -\sqrt{t^2+1}$ as this would give $y(0) = -1$.

Question 3.

(a) To find out if $f(x,y,z) = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2$ is convex, we compute its first order partial derivatives

$$f'_x = -4x + 2y - 2z, \quad f'_y = 2x - 4y + 4z, \quad f'_z = -2x + 4y - 4z$$

and its Hessian matrix

$$H(f) = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}$$

The leading principal minors are $D_1 = -4$, $D_2 = 12$ and $D_3 = -4 \cdot 12 - 4(-12) - 2(0) = 0$. We have used cofactor expansion along the last row to compute D_3. We see that the Hessian $H(f)$ may be negative semidefinite, and we must check if all principal minors $\Delta_i \geq 0$ to verify this. We compute that $\Delta_1 = -4, -4, -4 < 0$, $\Delta_2 = 12, 12, 0 \geq 0$ and $\Delta_3 = 0$. Hence $H(f)$ is negative semidefinite, and f is concave. Since $D_1 < 0$, $H(f)$ is not positive semidefinite, so f is not convex.

(b) The stationary points of f are the solutions of the first order conditions, given by

$$f'_x = -4x + 2y - 2z = 0, \quad f'_y = 2x - 4y + 4z = 0, \quad f'_z = -2x + 4y - 4z = 0$$

This is a linear system, and we solve it using Gassian elimination:

$$\begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

We have divided the last row by 2, to simplify computations. We see that z is a free variable, that $y = z$ and that $-2x = -y + z = 0$, which gives $x = 0$. Therefore, there are infinitely many stationary points, given by $(x, y, z) = (0, z, z)$. Since f is concave, all these stationary points are maxima, with maximum value $f(0, z, z) = -3$.

(c) Since $g(w) = 6/w$ has derivative $g'(w) = -6/w^2$ and $w = f(x, y, z) \le -3$ is negative (since -3 is the maximal value of f), it follows that g is a decreasing function of w. Therefore, the maximum of f gives a minimum of g. It follows that the minimum of g is obtained at the points $(0, z, z)$ with minimum value $g(0, z, z) = 6/f(0, z, z) = 6/(-3) = -2$. The function g has no maximum value, as this would correspond to minimum values of f, which does not exist. For example, when $y = z = 0$, we have that

$$g(x, 0, 0) = \frac{6}{f(x, 0, 0)} = \frac{6}{-3 - x^2} \rightarrow 0 \quad \text{when } x \rightarrow \infty$$

Hence $g(x, y, z) < 0$ for all x, y, z, and g takes values arbitrary close to 0, and g does not have a maximal value.

Question 4.

(a) The standard form of the Kuhn-Tucker problem is

$$\max \; -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 \text{ subject to } -x - y + z \le -2$$

The Lagrangian is $\mathcal{L} = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 - \lambda(-x - y + z)$. The first order conditions (FOC) are

$$\begin{aligned} \mathcal{L}'_x &= -4x + 2y - 2z + \lambda = 0 \\ \mathcal{L}'_y &= 2x - 4y + 4z + \lambda = 0 \\ \mathcal{L}'_y &= -2x + 4y - 4z - \lambda = 0 \end{aligned}$$

the constraint (C) is given by $x + y - z \ge 2$, and the complementary slackness conditions are given by $\lambda \ge 0$ and $\lambda(x + y - z - 2) = 0$.

(b) To find candidate points for maximum, we solve the Kuhn-Tucker conditions. In the case $x + y - z > 2$, we have that $\lambda = 0$, and this gives the linear system

$$\begin{aligned} -4x + 2y - 2z &= 0 \\ 2x - 4y + 4z &= 0 \\ -2x + 4y - 4z &= 0 \end{aligned}$$

with solution $(x, y, z) = (0, z, z)$ with z as a free variable from Question 3. At these points, $x + y - z = 0 + z - z = 0$, so the constraint $x + y - z > 2$ does not hold. There are no candidate points in this case. In the case when $x + y - z = 2$, we get the linear system

$$\begin{aligned} x + y - z &= 2 \\ -4x + 2y - 2z + \lambda &= 0 \\ 2x - 4y + 4z + \lambda &= 0 \\ -2x + 4y - 4z - \lambda &= 0 \end{aligned}$$

We use Gaussian elimination to solve this system, and get

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ -4 & 2 & -2 & 1 & 0 \\ 2 & -4 & 4 & 1 & 0 \\ -2 & 4 & -4 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & 6 & -6 & 1 & 8 \\ 0 & -6 & 6 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & 6 & -6 & 1 & 8 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This shows that there are infinitely many solutions to the Kuhn-Tucker conditions, with z free. Moreover, we see that $\lambda = 2$, that $6y = 6z - 2 + 8 = 6z + 6$, which gives $y = z + 1$, and that $x = -y + z + 2 = -(z + 1) + z + 2 = 1$. In other words, the solutions are

$$(x, y, z; \lambda) = (1, z + 1, z; 2)$$

for any value of z. We choose one of the these points, for example the point $(1, 1, 0; 2)$ with $z = 0$, and use the SOC: The Lagrangian

$$\mathcal{L}(x, y, z; 2) = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 - 2(-x - y + z)$$

has the same Hessian matrix as f in Question 3a). It follows that $\mathcal{L}$ is a concave function, and therefore $(1, 1, 0)$ is a maximum point with maximum value $f(1, 1, 0) = -5$. Any of the other solutions $(x, y, z) = (1, z + 1, z)$ of the Kuhn-Tucker conditions is also a maximum point with $f(1, z + 1, z) = -5$, since it gives the same Lagrangian.

(c) We consider the Kuhn-Tucker problem $\max f(x, y, z)$ subject to $ax + y - z \geq 2$, with constraint $-ax - y + z \leq -2$ in standard form. We know by b) that for $a = 1$, it has maximum value $f^*(1) = -5$ obtained at the point $(x^*(1), y^*(1), z^*(1); \lambda^*(1)) = (1, z + 1, z; 2)$. The Lagrangian of the Kuhn-Tucker problem with parameter a is

$$\mathcal{L} = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 - \lambda(-ax - y + z)$$

with $\partial \mathcal{L}/\partial a = \lambda x$. The relevant Envelope Theorem is that

$$\frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = \mathcal{L}'_a(x^*(a), y^*(a), z^*(a); \lambda^*(a)) = \lambda^*(a) \cdot x^*(a)$$

for values of the parameter a such that a maximum is obtained at $(x^*(a), y^*(a), z^*(a); \lambda^*(a))$, a solution of the Kuhn-Tucker conditions. This is the case for all values of a close to 1, and it implies that

$$f^*(1.12) \cong f^*(1) + 2 \cdot \Delta a = -5 + 2 \cdot (0.12) = -5 + 0.24 = -4.76$$

by the Envelope Theorem since $\lambda^*(1) \cdot x^*(1) = 2 \cdot 1 = 2$. In fact, for all $a > -1$, we find the solutions of the Kuhn-Tucker conditions by replacing the linear system in b) with the linear system

$$\left(\begin{array}{cccc|c} a & 1 & -1 & 0 & 2 \\ -4 & 2 & -2 & 1 & 0 \\ 2 & -4 & 4 & 1 & 0 \\ -2 & 4 & -4 & -1 & 0 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{cccc|c} -2 & 4 & -4 & -1 & 0 \\ -4 & 2 & -2 & 1 & 0 \\ 2 & -4 & 4 & 1 & 0 \\ a & 1 & -1 & 0 & 2 \end{array}\right)$$

We have swapped the first and last row, and this gives the echelon form

$$\rightarrow \left(\begin{array}{cccc|c} -2 & 4 & -4 & -1 & 0 \\ 0 & -6 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1+2a & -1-2a & -a/2 & 2 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{cccc|c} -2 & 4 & -4 & -1 & 0 \\ 0 & -1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1+a & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We see that there is an infinite number of solutions $(x^*(a), y^*(a), z^*(a); \lambda^*(a))$ with $\lambda^*(a) > 0$, and with z as a free variable. By the SOC, these solutions are maxima since the Lagrangian

$$\mathcal{L} = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2 - \frac{4}{1+a}(-ax - y + z)$$

is a concave function (it has the same Hessian as the Lagrangian in b).

Question 5.

To compute the eigenvalues of A, we solve the characteristic equation given by

$$|A - \lambda I| = \begin{vmatrix} -\alpha_2 - \lambda & \alpha_1 & 0 \\ -\alpha_3 & -\lambda & \alpha_1 \\ 0 & -\alpha_3 & \alpha_2 - \lambda \end{vmatrix} = 0$$

The determinant on the right hand side can be computed using cofactor expansion along the first column:

$$\begin{aligned}
|A-\lambda I| &= (-\alpha_2-\lambda)(-\lambda(\alpha_2-\lambda)+\alpha_1\alpha_3)+\alpha_3(\alpha_1(\alpha_2-\lambda)) \\
&= -(\lambda+\alpha_2)(\lambda^2-\alpha_2\lambda+\alpha_1\alpha_3)+\alpha_1\alpha_2\alpha_3-\alpha_1\alpha_3\lambda \\
&= -\lambda^3+(\alpha_2^2-2\alpha_1\alpha_3)\lambda = -\lambda(\lambda^2-(\alpha_2^2-2\alpha_1\alpha_3))
\end{aligned}$$

Therefore, the characteristic equation gives that $\lambda = 0$ or that $\lambda^2 = \alpha_2^2 - 2\alpha_1\alpha_3$. Let us write

$$\Delta = \alpha_2^2 - 2\alpha_1\alpha_3$$

If $\Delta > 0$, then $\lambda = 0$ and $\lambda = \pm\sqrt{\Delta}$ are the three distinct eigenvalues of A. In this case A is diagonalizable. If $\Delta < 0$, then $\lambda = 0$ is the only eigenvalue of A (with multiplicity one) and A is not diagonalizable. If $\Delta = 0$, then $\lambda = 0$ is the only eigenvalue of A, with multiplicity three, and A is only diagonalizable if $A = 0$, that is, if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. We conclude that A is diagonalizable if and only if $\alpha_2^2 - 2\alpha_1\alpha_3 > 0$ or $\alpha_1 = \alpha_2 = \alpha_3 = 0$.