Solutions	Final exam in GRA 6035 Mathematics
Date	June 20th, 2017 at 0900 - 1200

Question 1.

(a) When $a = -5$, the determinant of A is

$$\det(A) = \begin{vmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{vmatrix} = -4(16-4) - 2(-8-4) + 2(4+8) = -48 + 24 + 24 = 0$$

This means that $\operatorname{rk} A < 3$, and since there is a non-zero 2-minor, for example

$$D_2 = \begin{vmatrix} -4 & 2 \\ 2 & -4 \end{vmatrix} = 12 \neq 0$$

it follows that $\operatorname{rk}(A) = 2$.

(b) When $a = -5$, the linear system $A\mathbf{x} = \mathbf{0}$ has $3 - \operatorname{rk}(A) = 3 - 2 = 1$ free variables. Therefore, the solutions can be written in the form $\operatorname{span}(\mathbf{v})$ for a single vector $\mathbf{v}$. We compute the solutions using Gaussian elimination:

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 2 & -4 & 2 \\ -4 & 2 & 2 \\ 2 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 2 & -4 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & -4 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

We have not written the last column corresponding to $\mathbf{b} = \mathbf{0}$. The echelon form gives that z is free, that $-6y + 6z = 0$, or $y = z$, and that $2x - 4y + 2z = 2x - 2z = 0$, or $x = z$. The solutions to the linear system can be written in the form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \cdot \mathbf{v} \quad \text{with} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) We have that $\operatorname{rk}(A) < 3$ if and only if $\det(A) = 0$, and the determinant of A is given by

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1+a & 2 & 2 \\ 2 & 1+a & 2 \\ 2 & 2 & 1+a \end{vmatrix} \\ &= (1+a)((1+a)^2 - 4) - 2(2(1+a) - 4) + 2(4 - 2(1+a)) \\ &= (1+a)(a^2 + 2a - 3) - 2(2a - 2) + 2(2 - 2a) \\ &= (1+a)(a+3)(a-1) - 4(a-1) - 4(a-1) = (a-1)((a+1)(a+3) - 8) \\ &= (a-1)(a^2 + 4a - 5) = (a-1)(a-1)(a+5) = (a-1)^2 \cdot (a+5) \end{aligned}$$

Hence $\operatorname{rk}(A) < 3$ for $a = 1$ and $a = -5$, and $\operatorname{rk} A = 3$ otherwise. We have that $\operatorname{rk}(A) = 2$ when $a = -5$ from (a) and we see that $\operatorname{rk}(A) = 1$ for $a = 1$ since all three rows in A are equal in this case. It follows that $\operatorname{rk}(A) = 2$ if and only if $a = -5$.

(d) Since A is symmetric, it is diagonalizable, and we find D by computing the eigenvalues of A, given by the characteristic equation $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 1+a-\lambda & 2 & 2 \\ 2 & 1+a-\lambda & 2 \\ 2 & 2 & 1+a-\lambda \end{vmatrix} = 0$$

Notice that the determinant on the left-hand side is the determinant of the matrix A with parameter $b = a - \lambda$ instead of a. The computation of $\det(A)$ in (a) therefore gives the characteristic equation

$$(b-1)^2 \cdot (b+5) = 0 \quad \Leftrightarrow \quad (a - \lambda - 1)^2 (a - \lambda + 5) = 0$$

Hence the eigenvalues are given by $a-\lambda-1=0$, or $\lambda=a-1$, and $a-\lambda+5=0$, or $\lambda=a+5$. The eigenvalue $\lambda=a-1$ has multiplicity two. It follows that

$$D=\begin{pmatrix} a-1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & a+5 \end{pmatrix}$$

Question 2.

(a) The differential equation $y''+y'-6y=36t$ is second order linear, with solution $y=y_h+y_p$. The homogeneous equation $y''+y'-6y=0$ has characteristic equation $r^2+r-6=0$, with solutions $r=2$ and $r=-3$. Therefore, the general solution is

$$y_h=C_1\cdot e^{2t}+C_2\cdot e^{-3t}$$

Since the right hand side $f(t)=36t$ is linear, we guess a linear solution $y_p=At+B$, which gives $y'=A$ and $y''=0$. This gives $A-6(At+B)=36t$, or $-6A=36$ and $A-6B=0$, with solution $A=-6$ and $B=-1$. Therefore, the general solution is

$$y=y_h+y_p=C_1\cdot e^{2t}+C_2\cdot e^{-3t}-6t-1$$

(b) The differential equation $ty'-y=\ln(t)$ is linear, since it can be written $y'-(1/t)y=\ln(t)/t$. We solve it as a linear differential equation and find the integration factor u given by

$$\int -\frac{1}{t}\,\mathrm{d}t=-\ln(t)+C \quad\Rightarrow\quad u=e^{-\ln t}=e^{\ln(t^{-1})}=t^{-1}=1/t$$

Multiplication with u in the differential equation gives

$$(uy)'=\frac{\ln t}{t}\cdot u=\frac{\ln t}{t^2} \quad\Rightarrow\quad y=\frac{1}{u}\cdot\int\frac{\ln t}{t^2}\,\mathrm{d}t=t\cdot\int\frac{\ln t}{t^2}\,\mathrm{d}t$$

We use integration by parts with $u'=1/t^2$ and $v=\ln t$ to compute the last integral, and get

$$\int\frac{\ln t}{t^2}\,\mathrm{d}t=-\frac{1}{t}\ln(t)-\int\left(-\frac{1}{t}\right)\cdot\frac{1}{t}\,\mathrm{d}t=-\frac{\ln t}{t}-\frac{1}{t}+C$$

This implies that the general solution is

$$y=t\left(-\frac{\ln t}{t}-\frac{1}{t}+C\right)=-\ln t-1+Ct$$

(c) We try to solve the differential equation as an exact differential equation, and rewrite it to the form

$$p(t,y)+q(t,y)\cdot y'=\left(\frac{y-2t}{ty-t^2}-1\right)+\left(\frac{t}{ty-t^2}\right)\cdot y'=0$$

and try to find a function $h=h(t,y)$ such that $h'_t=p$ and $h'_y=q$. Since $(ty-t^2)'_t=y-2t$ and $(ty-t^2)'_y=t$, we see that

$$h(t,y)=\ln(ty-t^2)-t$$

is a solution, and the equation is exact. This gives

$$h(t,y)=\ln(ty-t^2)-t=C \quad\Rightarrow\quad \ln(yt-t^2)=C+t \quad\Rightarrow\quad yt-t^2=e^{C+t}=e^C\cdot e^t$$

With $K=e^C$, this gives $yt=Ke^t+t^2$, and therefore that

$$y=\frac{Ke^t+t^2}{t}=\frac{Ke^t}{t}+t$$

is the general solution of the differential equation. Finally, we check if the differential equation is linear, and multiply it with the denominator $ty-t^2$ and rewrite it as

$$(y-2t)+ty'=ty-t^2 \quad\Rightarrow\quad ty'+(1-t)y=2t-t^2 \quad\Rightarrow\quad y'+\frac{1-t}{t}\cdot y=\frac{2t-t^2}{t}$$

This shows that the equation is linear. It is possible to solve it as a linear equations, using the integration factor $u=te^{-t}$, and we would obtain the same result as above.

Question 3.

(a) To determine whether $f(x,y,z) = 9 - x^2 - y^2 - z^2 + 2xz$ is concave, we compute its first order partial derivatives

$$f'_x = -2x + 2z, \quad f'_y = -2y, \quad f'_z = -2z + 2x$$

and its Hessian matrix

$$H(f) = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

The leading principal minors are $D_1 = -2$, $D_2 = 4$ and $D_3 = 0$. We have used cofactor expansion along the middle row to compute D_3. We see that the Hessian $H(f)$ may be negative semidefinite, and we must check all principal minors Δ_i to verify this. We compute that $\Delta_1 = -2, -2, -2 < 0$, $\Delta_2 = 4, 0, 4 \geq 0$ and $\Delta_3 = 0$. Hence $H(f)$ is negative semidefinite, and f is concave. Any stationary point is therefore a global maximum point, and we see from the first order conditions that $(x,y,z) = (0,0,0)$ is one stationary point. In fact, we can solve the FOC's, and find that $y = 0$ and $x = z$, so all points of the form $(z, 0, z)$ are stationary points. The maximum value of f is $f(0,0,0) = 9$.

(b) We have that $g(x,y,z) = \ln(10 - f(x,y,z)) = \ln(1 + x^2 + y^2 + z^2 - 2xz) = \ln(u)$ with $u = 1 + x^2 + y^2 + z^2 - 2xz$. That stationary points of g is given by

$$\begin{aligned} g'_x &= \frac{1}{u} \cdot u'_x = \frac{1}{u} \cdot (2x - 2z) = 0 \\ g'_y &= \frac{1}{u} \cdot u'_y = \frac{1}{u} \cdot (2y) = 0 \\ g'_z &= \frac{1}{u} \cdot u'_z = \frac{1}{u} \cdot (2z - 2x) = 0 \end{aligned}$$

Since $f_{\max} = 9$, we have that $f(x,y,z) \leq 9$ and therefore that $u(x,y,z) \geq 10 - 9 = 1$. This means that $1/u > 0$, and the stationary points are given by $x = z$ and $y = 0$. Therefore, the stationary points of g are the points $(x,y,z) = (z, 0, z)$ with z free.

(c) Since $u \geq 1$ and $\ln(u)$ is an increasing function, it follows that $g(x,y,z) = \ln(u) \geq \ln(1) = 0$, and $g(z,0,z) = 0$ for the stationary points of g. Hence $w = 0$ is the minimal value of g. To determine whether g has a maximal value, let for instance $x = z = 0$. Then

$$g(x,y,z) = \ln(1 + x^2 + y^2 + z^2 - 2xz) = \ln(1 + y^2) \to \infty$$

when $y \to \infty$. Therefore, g has no maximal value, and the interval of possible values of g is given by $V_g = [0, \infty)$.

Question 4.

(a) The standard form of the Kuhn-Tucker problem is

$$\max\ 9 - x^2 - y^2 - z^2 + 2xz \text{ subject to } -x - y + z \leq -2$$

The Lagrangian is $\mathcal{L} = 9 - x^2 - y^2 - z^2 + 2xz - \lambda(-x - y + z)$. The first order conditions (FOC) are

$$\begin{aligned} \mathcal{L}'_x &= -2x + 2z + \lambda = 0 \\ \mathcal{L}'_y &= -2y + \lambda = 0 \\ \mathcal{L}'_y &= 2x - 2z - \lambda = 0 \end{aligned}$$

the constraint (C) is given by $x + y - z \geq 2$, and the complementary slackness conditions are given by $\lambda \geq 0$ and $\lambda(x + y - z - 2) = 0$.

(b) To find candidate points for maximum, we solve the Kuhn-Tucker conditions. In the case $x + y - z > 2$, we have that $\lambda = 0$, and from the FOC's, this gives $y = 0$ and $x = z$. The constraint $x + y - z > 2$ gives $0 > 2$, a contraction, so there are no candidate points with $x + y - z > 2$. If $x + y - z = 2$, then the FOC's give $\lambda = 2y$ (from the middle one) and

$2x - 2y + 2z = 0$ (from the first and last). Combined with the constraint, this gives the linear system

$$\begin{aligned} x + y - z &= 2 \\ 2x - 2y + 2z &= 0 \end{aligned}$$

with solution $(x, y, z) = (1 + z, 1, z)$ with z as a free variable, using Gaussian elimination: We add -2 times the first row to the second to obtain an echelon form with $-4y = -4$, which gives $y = 1$, and $x = 1 + z$. This gives the candidate point $(x, y, z; \lambda) = (1 + z, 1, z; 2)$ with function value $f(1 + z, 1, z) = 7$. As f is concave from Question 3, it follows from the SOC that the maximum value is $f = 7$ (since $\mathcal{L}$ is also concave when the constraint is linear).

(c) If we replace f with f_a with $a > 0$, then f is still concave with Hessian matrix

$$H(f_a) = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2a & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

where $\Delta_1 = -2, -2a - 2 < 0$, $\Delta_2 = 4a, 0, 4a \geq 0$, $\Delta_3 = 0$. This implies that $\mathcal{L}_a = f_a - \lambda(x + y - z)$ is also concave, since the constraint is linear. There is still a candidate point where the constraint is binding, since we get the linear system

$$\begin{aligned} x + y - z &= 2 \\ 2x - 2ay + 2z &= 0 \end{aligned}$$

from the constraint and the FOC's, and this system has solutions. It follows from the SOC that the new Kuhn-Tucker problem has a maximum value $f^*(a)$ for $a > 0$. Using the Envelope Theorem, we get that

$$\frac{df^*(a)}{da} = \frac{\partial}{\partial a} \mathcal{L}_a^(x^*(a), y^*(a), z^*(a); \lambda^*(a)) = -y^*(a)^2$$

and this derivative is equal to -1 at $a = 1$ since $y^*(a) = 1$ from (b). We therefore estimate the new maximum value to be

$$f^*(1.25) \cong f^*(1) + 0.25 \cdot (-1) = 6.75$$

since $\Delta a = 1.25 - 1 = 0.25$.