Exam	Final exam in GRA 6035 Mathematics
Date	June 20th, 2017 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 1+a & 2 & 2 \\ 2 & 1+a & 2 \\ 2 & 2 & 1+a \end{pmatrix}$$

(a) Compute the determinant and rank of A when $a = -5$.
(b) When $a = -5$, find a vector $\mathbf{v}$ such that $\text{span}(\mathbf{v})$ is the set of solutions of $A \cdot \mathbf{x} = \mathbf{0}$.
(c) Determine all values of a such that $\text{rk}\, A = 2$.
(d) Find a diagonal matrix D such that $D = P^{-1}AP$ for an invertible matrix P.

Question 2.

We consider differential equations in the function $y = y(t)$.

(a) Solve the differential equation $y'' + y' - 6y = 36t$.
(b) Solve the differential equation $ty' - y = \ln(t)$.
(c) Show that the differential equation

$$\frac{y - 2t}{ty - t^2} + \frac{t}{ty - t^2} \cdot y' = 1$$

is both linear and exact, and solve it.

Question 3.

We consider the functions $f(x, y, z) = 9 - x^2 - y^2 - z^2 + 2xz$ and $g(x, y, z) = \ln(10 - f(x, y, z))$.

(a) Explain that f is concave, and find its maximum value.
(b) Find all stationary points of g.
(c) Determine whether g has a maximal and/or a minimal value. Give your answer as the interval of possible values $w = g(x, y, z)$ of g. It is not necessary to compute the Hessian matrix of g.

Question 4.

We consider the following Kuhn-Tucker problem:

$$\max\ f(x, y, z) = 9 - x^2 - y^2 - z^2 + 2xz \quad \text{subject to} \quad x + y - z \geq 2$$

(a) Write down all Kuhn-Tucker conditions for this problem.
(b) Solve the Kuhn-Tucker problem and find its maximum value.

Consider the function $f_a(x, y, z) = 9 - x^2 - ay^2 - z^2 + 2xz$ with parameter a, and the Kuhn-Tucker problem where the objective function f is replaced by f_a.

(c) Explain that the new Kuhn-Tucker problem has a maximum value when $a > 0$, and estimate this maximum value when $a = 1.25$.