Solutions	Final exam in GRA 6035 Mathematics
Date	January 18th, 2021 at 0900 - 1200

Question 1.

(a) The nullspace of A is the set of solutions of $A\mathbf{x} = \mathbf{0}$. We use Gaussian elimination to find the solutions:

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 6 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 22 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivot positions, we have that $\dim \mathrm{Null}(A) = 4 - 3 = 1$, with y free since there is no pivot position in the y column. To find solutions (x, y, z, w) we use back substitution. This gives $16w = 0$, or $w = 0$ from the third equation, $2z - w = 0$, or $z = 0$ from the second, and $x + 2y + 2w = 0$, or $x = -2y$ with y free from the first. The solutions are therefore given by

$$(x, y, z, w) = (-2y, y, 0, 0) = y \cdot (-2, 1, 0, 0) = y \cdot \mathbf{w}$$

where $\mathbf{w} = (-2, 1, 0, 0)$, and $\mathcal{B} = \{\mathbf{w}\}$ is a base of $\mathrm{Null}(A)$. To find the eigenvalues of A, we solve the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 & 2 \\ 2 & 4-\lambda & 2 & 0 \\ 0 & 0 & 5-\lambda & 6 \\ 0 & 0 & 6 & 10-\lambda \end{vmatrix} = 0$$

We use cofactor expansion along the first column to compute the determinant, and get

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda) \cdot (4-\lambda) \begin{vmatrix} 5-\lambda & 6 \\ 6 & 10-\lambda \end{vmatrix} - 2 \cdot 2 \begin{vmatrix} 5-\lambda & 6 \\ 6 & 10-\lambda \end{vmatrix} \\ &= [(1-\lambda)(4-\lambda) - 4] \cdot [(5-\lambda)(10-\lambda) - 36] \\ &= (\lambda^2 - 5\lambda)(\lambda^2 - 15\lambda + 14) = \lambda(\lambda - 5)(\lambda - 1)(\lambda - 14) \end{aligned}$$

Therefore the eigenvalues of A are $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 5,\ \lambda_4 = 14$.

(b) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the column vectors of A. We see from the echelon form above that there are pivot positions in column 1, 3 and 4 of A. Hence $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ is a base of $\mathrm{Col}(A)$. Since $\dim \mathrm{Col}(A) = 3 < 4 = \dim \mathbb{R}^4$, there are vectors in $\mathbb{R}^4$ not in $\mathrm{Col}(A)$. We use Gaussian elimination to find all vectors (a, b, c, d) that are in $\mathrm{Col}(A)$:

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & a \\ 2 & 4 & 2 & 0 & b \\ 0 & 0 & 5 & 6 & c \\ 0 & 0 & 6 & 10 & d \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & a \\ 0 & 0 & 2 & -4 & b-2a \\ 0 & 0 & 10 & 12 & 2c \\ 0 & 0 & 6 & 10 & d \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & a \\ 0 & 0 & 2 & -4 & b-2a \\ 0 & 0 & 0 & 32 & 2c - 5b + 10a \\ 0 & 0 & 0 & 22 & d - 3b + 6a \end{array}\right)$$

This linear system has solutions if and only if $32(d - 3b + 6a) = 22(2c - 5b + 10a)$, which can be written $-28a + 14b - 44c + 32d = 0$, since a non-zero determinant correspond to a pivot position in the fourth and fifth column. We can write this equation as $-14a + 7b - 22c + 16d = 0$, and this means that $\mathbf{v} = (a, b, c, d)$ is not in $\mathrm{Col}(A)$ if and only if $-14a + 7b - 22c + 16d \neq 0$. One example of a vector not in $\mathrm{Col}(A)$ is $\mathbf{v} = (1, 0, 0, 0)$ since $-14 \cdot 1 = -14 \neq 0$.

(c) The symmetric matrix B of the quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\, \mathbf{x}$ is given by

$$B = \frac{1}{2}\left(A + A^T\right) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \\ 0 & 1 & 5 & 6 \\ 1 & 0 & 6 & 10 \end{pmatrix}$$

and the first three leading principal minors of B are $D_1 = 1$, $D_2 = 1 \cdot 4 - 2 \cdot 2 = 0$, and

$$D_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 5 \end{vmatrix} = 1(20 - 1) - 2(10 - 0) = 19 - 20 = -1$$

Since $D_1 > 0$ and $D_3 < 0$, the quadratic form f in indefinite.

(d) Let us write $\mathbf{x} = (x, y, z, w)$ for the variables. Then the constraint is $x^2+y^2+z^2+w^2 = 1$ since $\mathbf{x}^T\mathbf{x} = \mathbf{x}^T I\mathbf{x}$. Hence the set of admissible points is closed and bounded, and the Lagrange problem has a solution. It is also clear that there are no admissible points where NDCQ fails, since

$$\operatorname{rk} J = \operatorname{rk}\begin{pmatrix} 2x & 2y & 2z & 2w \end{pmatrix} = \begin{cases} 0, & (x,y,z,w) = (0,0,0,0) \\ 1, & (x,y,z,w) \neq (0,0,0,0) \end{cases}$$

and $(0,0,0,0)$ is clearly not admissible since $\mathbf{x}^T\mathbf{x} = 0 \neq 1$ at this point. Hence the problem has a max and a min among the ordinary candidate points $(\mathbf{x};\lambda)$ where the first order conditions hold. The Lagrangian is $\mathcal{L} = f(\mathbf{x}) - \lambda(\mathbf{x}^T\mathbf{x})$, and the first order conditions can be written in matrix form as

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = 2B\mathbf{x} - \lambda \cdot 2I\mathbf{x} = 2(B - \lambda I)\mathbf{x} = \mathbf{0}$$

where B is the symmetric matrix of the quadratic form f. We could also find the first order conditions by writing out the quadratic form f and the constraint. This implies that $\mathbf{x} = \mathbf{0}$ or that $\mathbf{x}$ is an eigenvector of B with eigenvalue λ. Clearly, $\mathbf{x} = \mathbf{0}$ does not satisfy the constraint. In fact, the constraint $\mathbf{x}^T\mathbf{x} = 1$ means that $\mathbf{x}$ is a vector of length one. We notice that the matrix B has four eigenvalues counted with multiplicity since it is symmetric, and we can write them $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. For each eigenvalue λ, there are at least two eigenvectors of length one with eigenvalue λ. In fact, there are two if $\dim E_\lambda = 1$ and infinitely many otherwise. Based on this, we conclude that there are at least two candidate points $(\mathbf{x};\lambda)$ for each eigenvalue λ where $\mathbf{x}$ is an eigenvector of B with eigenvalue λ, and that these are the only candidate points. The value of f at such a candidate point is given by

$$f(\mathbf{x}) = \mathbf{x}^T B\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} = \lambda \cdot 1 = \lambda$$

Hence the maximum value in the Lagrange problem is given by $f_{\max} = \lambda_4$ (the maximal eigenvalue of B), and the minimum value is given by $f_{\min} = \lambda_1$ (the minimal eigenvalue of B).

Question 2.

(a) We have that $f(x,y,z) = \ln(5-u)$ where $u = u(x,y,z) = x^2+y^2+z^2-xy+xz-yz$ is a quadratic form. The symmetric matrix of the quadratic form is given by

$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix}$$

Since the leading principal minors are $D_1 = 1$, $D_2 = 1 - 1/4 = 3/4$, and

$$D_3 = |A| = 1(1-1/4) + 1/2(-1/2+1/4) + 1/2(1/4-1/2) = 3/4 - 1/8 - 1/8 = 1/2$$

with $D_1, D_2, D_3 > 0$, u is positive definite. This means that $u(x,y,z) \geq 0$, and therefore that $5 - u(x,y,z) \leq 5$ at all points (x,y,z). We conclude that $f_{\max} = \ln 5$ since $h(u) = \ln(u)$ is a strictly increasing function; that is, $u_1 < u_2$ implies that $h(u_1) < h(u_2)$ since $h'(u) = 1/u > 0$. The maximizer is $(x,y,z) = (0,0,0)$ with $f(0,0,0) = \ln 5$.

(b) A subset of $D \subseteq \mathbb{R}^4$ is compact if and only if it is closed and bounded, and by definition, it is bounded if there are finite numbers a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 such that

$$a_1 \leq x \leq b_1,\ a_2 \leq y \leq b_2,\ a_3 \leq z \leq b_3,\ a_4 \leq w \leq b_4$$

for all $(x,y,z,w) \in D$. Since the point $(a,a,-a,-a)$ give $xw + yz = -2a^2$, and $-2a^2 \leq -2$ when $a^2 \geq 1$, we have that $(a,a,-a,-a) \in D$ for all $a \geq 1$ and all $a \leq -1$. This means that D is not bounded, and therefore not compact. The set D is closed since it is given by a closed inequality.

(c) The Kuhn-Tucker problem can be written in standard form as

$$\max -f(x,y,z,w) = -x^2 - 4y^2 - 9z^2 - w^2 \text{ subject to } xw + yz \leq -2$$

It Lagrangian is $\mathcal{L} = -x^2 - 4y^2 - 9z^2 - w^2 - \lambda(xw + yz)$, and the first order conditions (FOC) are

$$\mathcal{L}'_x = -2x - \lambda w = 0 \qquad \mathcal{L}'_y = -8y - \lambda z = 0$$
$$\mathcal{L}'_z = -18z - \lambda y = 0 \qquad \mathcal{L}'_w = -2w - \lambda x = 0$$

The constraint (C) is given by $xw + yz \leq -2$, and the complementary slackness conditions (CSC) by $\lambda \geq 0$ and $\lambda(xw + yz + 2) = 0$. The first and last FOC's give

$$x = -\frac{1}{2}\lambda w \quad \Rightarrow \quad -2w = \lambda\left(-\frac{1}{2}\lambda w\right) = -\frac{1}{2}\lambda^2 w$$

This means that $w = \lambda^2 w/4$, which give $w = 0$ or $\lambda^2 = 4$. Since $\lambda \geq 0$ by the CSC, this implies that either $w = x = 0$, or $\lambda = 2$ and $x = -w$. In a similar way, the two middle FOC's give

$$y = -\frac{1}{8}\lambda z \quad \Rightarrow \quad -18z = \lambda\left(-\frac{1}{8}\lambda z\right) = -\frac{1}{8}\lambda^2 z$$

This means that $z = \lambda^2 z/144$, which give $z = 0$ or $\lambda^2 = 144$. Since $\lambda \geq 0$ by the CSC, this implies that either $z = y = 0$, or $\lambda = 12$ and $y = -3z/2$. From the constraint, we see that the point $(x, y, z, w) = (0, 0, 0, 0)$ is not admissible, and we are left with the following possibilities:

$$(x, y, z, w; \lambda) = (0, -3z/2, z, 0; 12),\ (-w, 0, 0, w; 2)$$

By the second part of the CSC, we have that $xw - yz = -2$ since $\lambda > 0$. In the first case, we get that $xw + yz = -3z/2 \cdot z = -3z^2/2 = -2$, or $z^2 = 4/3$. In the second case, we get that $xw + yz = -w \cdot w = -w^2 = -2$, or $w^2 = 2$. Hence the following points are the solutions of the Kuhn-Tucker conditions, with corresponding f-values:

$$(x, y, z, w; \lambda) = (0, -\sqrt{3}, 2/\sqrt{3}, 0; 12), (0, \sqrt{3}, -2/\sqrt{3}, 0; 12), \quad f = 4 \cdot 3 + 9 \cdot 4/3 = 24$$
$$= (-\sqrt{2}, 0, 0, \sqrt{2}; 2), (\sqrt{2}, 0, 0, -\sqrt{2}; 2), \quad f = 2 + 2 = 4$$

Question 3.

(a) We solve the second order linear difference equation $y_{t+2} - y_{t+1} - 2y_t = 4t$ using superposition. To find y_t^h, we consider the homogeneous equation $y_{t+2} - y_{t+1} - 2y_t = 0$, which has characteristic equation $r^2 - r - 2 = 0$ with roots $r = -1$ and $r = 2$. Hence $y_t^h = C_1(-1)^t + C_2 \cdot 2^t$. To find the particular solution y_t^p, we consider the difference equation $y_{t+2} - y_{t+1} - 2y_t = 4t$ and use the method of undetermined coefficients. Since $f_t = 4t$, we use $y_t = At + B$, which gives $y_{t+1} = A(t + 1) + B$ and $y_{t+2} = A(t + 2) + B$. When we substitute this into the difference equation, we get

$$A(t + 2) + B - (A(t + 1) + B) - 2(At + B) = 4t \quad \Leftrightarrow \quad -2At + (A - 2B) = 4t$$

Comparing coefficients, we get $-2A = 4$, or $A = -2$, and $A - 2B = 0$, or $B = A/2 = -1$. Hence $y_p = -2t - 1$ and the general solution is therefore

$$y = y_t^h + y_t^p = C_1(-1)^t + C_2 \cdot 2^t - 2t - 1$$

With initial conditions $y_0 = y_1 = 1$, we get $C_1 + C_2 - 1 = 1$, or $C_1 + C_2 = 2$ from the first condition, and $-C_1 + 2C_2 - 3 = 1$, or $-C_1 + 2C_2 = 4$ from the second. Adding these equations, we get $3C_2 = 6$, or $C_2 = 2$, and this gives $C_1 = 0$. The particular solution is therefore

$$y = 2 \cdot 2^t - 2t - 1 = 2^{t+1} - 2t - 1$$

This means that $y_{17} = 2^{18} - 2(17) - 1 = 262\,109$.

(b) The differential equation $t^2 y' + 2ty = 1$ can be written $t^2 y' = 1 - 2ty$ or $y' = (1 - 2ty)/t^2$. The right-hand side cannot be factored as $f(t) \cdot g(y)$, hence the equation is not separable. The differential equation can be written on the form $y' + a(t)y = b(t)$, since division by t^2 gives

$$y' + \frac{2}{t}y = \frac{1}{t^2}$$

Hence the equation is linear. We solve it using integrating factor: The integrating factor is $u = e^{2\ln t} = (e^{\ln t})^2 = t^2$ since $\int 2/t \, \mathrm{d}t = 2\ln|t| + C$ and $|t|^2 = t^2$. Multiplying with $u = t^2$, we get

$$t^2y' + 2ty = 1 \quad \Leftrightarrow \quad (t^2y)' = 1 \quad \Leftrightarrow \quad t^2y = \int 1 \, \mathrm{d}t = t + C$$

The equation has general solution $y = 1/t + C/t^2$. Finally, the differential equation can be written $t^2y' + 2ty - 1 = 0$, and with $h = t^2y - t$ we have

$$t^2y' + (2ty - 1) = h'_y \cdot y' + h'_t = 0$$

Therefore the equation is exact, and the solution is given by $h(t, y) = C$. This gives

$$t^2y - t = C \quad \Leftrightarrow \quad y = \frac{C + t}{t^2} = C/t^2 + 1/t$$

We notice that this is the same solution as we found above using integrating factor.

(c) We can write $y = 3e^{-2t} - 5e^t + 12e^{-3t} = y_h + y_p$ in several different ways. If the genaral homogeneous solution is $y_h = C_1e^{-2t} - C_2e^t$, then the characteristic roots are $r = -2$ and $r = 1$, which gives the characteristic equation $(r + 2)(r - 1) = r^2 + r - 2 = 0$. Hence the homogeneous equation is $y'' + y' - 2y = 0$. If we substitute $y = y_p = 12e^{-3t}$ in the left-hand side, we get

$$y'' + y' - 2y = (-3)^2 \cdot 12e^{-3t} + (-3) \cdot 12e^{-3t} - 2 \cdot 12e^{-3t} = 12e^{-3t}(9 - 3 - 2) = 48e^{-3t}$$

It follows that the linear second order differential equation $y'' + y' - 2y = 48e^{-3t}$ has y as a particular solution. Using the other decompositions $y = y_h + y_p$, we find that the linear second order differential equations

$$y'' + 5y' + 6y = -60\, e^t \quad \text{or} \quad y'' + 2y' - 3y = -9\, e^{-2t}$$

also have y as a particular solution.

(d) We consider the system $\mathbf{y}' = A\mathbf{y}$ with $\mathbf{y} = (y, y', y'')$. It has the form

$$\mathbf{y}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix} \cdot \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = A\mathbf{y}$$

where the coefficients in the first and second row follows from the fact that $y_1' = y_2$ and $y_2' = y_3$ when $(y_1, y_2, y_3) = (y, y', y'')$. We know that when A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then the general solution of the system is given by

$$\mathbf{y} = C_1\mathbf{v}_1e^{\lambda_1 t} + C_2\mathbf{v}_2e^{\lambda_2 t} + C_3\mathbf{v}_3e^{\lambda_3 t}$$

Comparing with $y = 3e^{-2t} - 5e^t + 12e^{-3t}$ from (c), we see that we should try to find a diagonalizable matrix A of the above form with eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = -3$. We use this to determine r, s, t in the last row of A. In fact, the eigenvalues of A are given by the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ r & s & t - \lambda \end{vmatrix} = 0$$

Cofactor expansion along the first row gives

$$0 = -\lambda(-\lambda(t - \lambda) - s) - 1(0 - r) = -\lambda(\lambda^2 - t\lambda - s) + r = -\lambda^3 + t\lambda^2 + s\lambda + r$$

On the other hand, since the eigenvalues of A should be $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = -3$, the characteristic equation is given by

$$-(\lambda + 2)(\lambda - 1)(\lambda + 3) = -(\lambda^2 + \lambda - 2)(\lambda + 3) = -(\lambda^3 + 4\lambda^2 + \lambda - 6) = 0$$

This can be written $-\lambda^3 - 4\lambda^2 - \lambda + 6 = 0$. Comparing the two characteristic equations, we find that $t = -4$, $s = -1$, and $r = 6$. This means that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix}$$

We see that this matrix is diagonalizable with three distinct eigenvalues.