Solutions	Final exam in GRA 6035 Mathematics
Date	December 2nd, 2021 at 0900 - 1200

Question 1.

(a) We use Gaussian elimination to find the rank of A:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -6 & -4 \\ 0 & 2 & -6 & -4 \\ 0 & -2 & 6 & 4 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -6 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are two pivot positions, we have that $\text{rk}(A) = 2$. Since the pivot positions are in the first two columns, the first two column vectors of A form a base $B = \{(1, 3, 4, 1), (0, 2, 2, -2)\}$ of $\text{Col}(A)$.

(b) We check if $\mathbf{v}$ is an eigenvector of A by computing $A\mathbf{v}$ and try to write the product as $\lambda\mathbf{v}$:

$$A\mathbf{v} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \\ -3 \end{pmatrix} = 0 \cdot \mathbf{v}$$

Since this is possible with $\lambda = 0$, it follows that $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$.

(c) We have that $\det(A) = 0$, since A is a 4×4 matrix with $\text{rk}(A) = 2 < 4$ from (a). An alternative argument is that since $\lambda = 0$ is an eigenvalue of A from (b), it is a solution of $|A - \lambda I| = 0$, and this means that $|A - 0 \cdot I| = 0$, or $|A| = 0$. It is also possible to compute $\det(A)$ directly, for instance using cofactor expansion.

(d) We have that $|S| = 1 \cdot 2 \cdot 4 = 8 \neq 0$, hence S has an inverse matrix S^{-1}. For any eigenvalue $\lambda \neq 0$ of S, there is an eigenvector $\mathbf{v}$ such that

$$S\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad \mathbf{v} = S^{-1}\lambda\mathbf{v} = \lambda S^{-1}\mathbf{v} \quad \Rightarrow \quad \frac{1}{\lambda} \cdot \mathbf{v} = S^{-1}\mathbf{v}$$

by multiplication with S^{-1} and $1/\lambda$. It follows that if $\lambda \neq 0$ is an eigenvalue of S, then $1/\lambda$ is an eigenvalue of S^{-1}. Hence the eigenvalues of S^{-1} are $1, 1/2, 1/4 > 0$, and it follows that S^{-1} is positive definite. We comment that since S is symmetric and invertible, the inverse S^{-1} is symmetric, so that the definiteness of S^{-1} is well-defined. This is implicit in the question and not necessary to prove, but we include an argument: We have that

$$S^{-1} = \frac{1}{8} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{3n} \end{pmatrix}^T$$

where C_{ij} are the cofactors of S. Since S is symmetric, it follows that $C_{ij} = C_{ji}$, and S^{-1} is therefore also a symmetric matrix.

Question 2.

(a) We use superposition to solve the linear differential equation $y'' + y' = 6e^{3t}$: To find the homogeneous solution y_h, we consider the homogeneous differential equation $y'' + y' = 0$ with characteristic equation $r^2 + r = 0$. It has two distinct solutions $r = 0$, $r = -1$, and therefore

$$y_h = C_1\, e^0 + C_2\, e^{-t} = C_1 + C_2\, e^{-t}$$

To find a particular solution y_p, we consider the differential equation $y'' + y' = 6e^{3t}$ and use the method of undetermined coefficients: We try to find solutions of the form $y = Ae^{3t}$, which gives $y' = 3Ae^{3t}$ and $y'' = 9Ae^{3t}$. When we substitute this into the differential equation, we get $(9Ae^{3t}) + 3(Ae^{3t}) = 6e^{3t}$, which gives $12A = 6$, or $A = 1/2$. The general solution of the differential equation is therefore

$$y = y_h + y_p = C_1 + C_2\, e^{-t} + \frac{1}{2}\, e^{3t}$$

(b) The differential equation $t(y'-y)=y$ can be written $y'-y=y/t$, or $y'=y+y/t=y(1+1/t)$, and it is both linear and separable. We choose to solve it as a separable differential equation:

$$y'=y\left(1+\frac{1}{t}\right) \quad\Rightarrow\quad \frac{1}{y}\,y'=1+\frac{1}{t} \quad\Rightarrow\quad \int\frac{1}{y}\,\mathrm{d}y=\int\left(1+\frac{1}{t}\right)\,\mathrm{d}t$$

This gives $\ln|y|=t+\ln|t|+C$, or $|y|=e^{t+\ln|t|+C}=e^t\cdot|t|\cdot e^C$. We therefore find the general solution $y=K\,te^t$ with $K=\pm e^C$.

(c) We use superposition to solve the linear difference equation $y_{t+2}+3y_{t+1}-4y_t=5$: To find the homogeneous solution y_t^h, we consider the characteristic equation $r^2+3r-4=0$, with two distinct roots $r=1,\ r=-4$, and therefore

$$y_t^h=C_1\cdot 1^t+C_2\cdot(-4)^t=C_1+C_2\cdot(-4)^t$$

To find a particular solution y_t^p, we consider the difference equation $y_{t+2}+3y_{t+1}-4y_t=5$. We try to find a constant solution $y_t=A$, which gives $y_{t+1}=y_{t+2}=A$. When we substitute this into the difference equation, we get $A+3A-4A=5$, or $0\cdot A=5$, which has no solutions. Next, we try to find solutions of the form $y_t=A\cdot t=At$, which gives $y_{t+1}=A(t+1)=At+A$, and $y_{t+2}=A(t+2)=At+2A$. When we substitute this into the difference equation, we get

$$(At+2A)+3(At+A)-4(At)=5 \quad\Rightarrow\quad 5A=5$$

We find the solution $A=1$, or $y_t^p=t$. The general solution is therefore given by

$$y_t=y_t^h+y_t^p=C_1+C_2\cdot(-4)^t+t$$

(d) We let A be the 3×3 matrix such that the system of differential equations can be written in the form $\mathbf{y}'=A\mathbf{y}$. The eigenvalues of A is given the characteristic equation

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 1 & 2\\ -1 & -\lambda & 1\\ 0 & 1 & 3-\lambda\end{vmatrix}=0$$

We use cofactor expansion along the first column to compute the determinant, and get

$$(1-\lambda)(-\lambda(3-\lambda)-1)-(-1)((3-\lambda)-2)=(1-\lambda)(\lambda^2-3\lambda-1)+(1-\lambda)$$

We see that $1-\lambda$ is a common factor, and write the characteristic equation in factorized form $(1-\lambda)(\lambda^2-3\lambda)=\lambda(1-\lambda)(\lambda-3)=0$. This gives three distinct eigenvalues $\lambda_1=0$, $\lambda_2=1$, and $\lambda_3=3$, and this means that A is diagonalizable. We find a base $\{\mathbf{v}_i\}$ for E_{λ_i} in each case: We use the Gaussian processes

$$E_0:\begin{pmatrix}1 & 1 & 2\\ -1 & 0 & 1\\ 0 & 1 & 3\end{pmatrix}\to\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 3\\ 0 & 0 & 0\end{pmatrix} \qquad E_1:\begin{pmatrix}0 & 1 & 2\\ -1 & -1 & 1\\ 0 & 1 & 2\end{pmatrix}\to\begin{pmatrix}-1 & -1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$E_3:\begin{pmatrix}-2 & 1 & 2\\ -1 & -3 & 1\\ 0 & 1 & 0\end{pmatrix}\to\begin{pmatrix}-1 & -3 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and back substitution, and find base vectors $\mathbf{v}_1=(1,-3,1),\ \mathbf{v}_2=(3,-2,1),\ \mathbf{v}_3=(1,0,1)$ for the three eigenspaces. The general solution is therefore given by

$$\mathbf{y}=C_1\mathbf{v}_1e^{\lambda_1 t}+C_2\mathbf{v}_2e^{\lambda_2 t}+C_3\mathbf{v}_3e^{\lambda_3 t}=C_1\begin{pmatrix}1\\-3\\1\end{pmatrix}+C_2\begin{pmatrix}3\\-2\\1\end{pmatrix}e^t+C_3\begin{pmatrix}1\\0\\1\end{pmatrix}e^{3t}$$

The initial condition $\mathbf{y}(0)=(5,-5,3)$ gives the linear system $C_1\mathbf{v}_1+C_2\mathbf{v}_2+C_3\mathbf{v}_3=\mathbf{y}(0)$, or $P\cdot\mathbf{C}=\mathbf{y}(0)$, where $P=(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$ and $\mathbf{C}$ is the column vector given by $\mathbf{C}=(C_1,C_2,C_3)$. We solve this using Gaussian elimination:

$$\left(\begin{array}{ccc|c}1 & 3 & 1 & 5\\ -3 & -2 & 0 & -5\\ 1 & 1 & 1 & 3\end{array}\right)\to\left(\begin{array}{ccc|c}1 & 3 & 1 & 5\\ 0 & 7 & 3 & 10\\ 0 & -2 & 0 & -2\end{array}\right)\to\left(\begin{array}{ccc|c}1 & 3 & 1 & 5\\ 0 & -2 & 0 & -2\\ 0 & 0 & 3 & 3\end{array}\right)$$

Back substitution gives $C_1 = C_2 = C_3 = 1$, and we find the particular solution

$$\mathbf{y} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

Question 3.

(a) We write down the symmetric matrix A of the quadratic form g and determine its definiteness: We find that

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$$

The leading principal minors are $D_1 = 3$, $D_2 = 3 - 1 = 2$, $D_3 = 3(3) - 1(-1) + 4(2-4) = 2$, and $D_4 = |A| = 2$, since the determinant is given by cofactor expansion along the last row:

$$D_4 = -(-1) \cdot \begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ 2 & 7 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \\ 4 & 7 & 0 \end{vmatrix} + 4 \cdot D_3 = -2 - 4 + 8 = 2$$

Since all leading principal minors are positive, g is a positive definite quadratic form.

(b) The Kuhn-Tucker problem is in standard form and has Lagrangian $\mathcal{L} = \mathbf{e}^T\mathbf{x} - \lambda(\mathbf{x}^T A\mathbf{x} - 18)$, where $\mathbf{e} = (1,1,1,1)$ is considered as a column vector. The first order conditions (FOC) can therefore be written $\mathbf{e} - \lambda \cdot 2A\mathbf{x} = \mathbf{0}$, the constraint (C) can be written $\mathbf{x}^T A\mathbf{x} \le 18$, and the complementary slackness conditions can be written $\lambda \ge 0$ and $\lambda(\mathbf{x}^T A\mathbf{x} - 18) = 0$. Together, the conditions FOC + C + CSC are the Kuhn-Tucker conditions of the problem:

$$\text{FOC+C+CSC:} \quad \mathbf{e} - 2\lambda A\mathbf{x} = \mathbf{0},\ \mathbf{x}^T A\mathbf{x} \le 18,\ \lambda \ge 0,\ \lambda(\mathbf{x}^T A\mathbf{x} - 18) = 0$$

(c) In case $g(\mathbf{x}) < 18$, there is no condition, and in case $g(\mathbf{x}) = 18$, the NDCQ is given by

$$\text{rk}\, J = \text{rk}\begin{pmatrix} g'_x & g'_y & g'_z & g'_w \end{pmatrix} = 1$$

This condition fails if and only if the rk $J = 0$, or $g'_x = g'_y = g'_z = g'_w = 0$. This is the condition for stationary points of g, and can be written $2A\mathbf{x} = \mathbf{0}$. Since $|A| = D_4 = 2 \neq 0$ from (a), A is invertible, and $\mathbf{x} = \mathbf{0}$ is the only stationary point of g. This point does not satisfy $g(\mathbf{x}) = 18$. We conclude that there are no admissible points where NDCQ does not hold.

(d) We see that if $\lambda = 0$, then the FOC's give $\mathbf{e} = \mathbf{0}$, or $(1,1,1,1) = (0,0,0,0)$, which is clearly impossible. By the CSC's, we must have that $\lambda > 0$ and that $g(\mathbf{x}) = 18$. To solve for candidate points in this case, we consider the FOC's, which give a linear system of equations:

$$2\lambda A\mathbf{x} = \mathbf{e} \quad \to \quad A\mathbf{x} = \frac{1}{2\lambda}\mathbf{e} = t\mathbf{e} = (t,t,t,t) \text{ with } t = \frac{1}{2\lambda}$$

We use Gauss to solve this linear system, and start by switching the first two rows:

$$\left(\begin{array}{rrrr|r} 3 & 1 & 4 & -1 & t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & t \\ 3 & 1 & 4 & -1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right)$$

We then use a standard Gaussian process:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & t \\ 3 & 1 & 4 & -1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & t \\ 0 & -2 & -2 & -4 & -2t \\ 0 & -2 & -1 & -4 & -3t \\ 0 & 2 & 2 & 5 & 2t \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 2 & 1 & t \\ 0 & -2 & -2 & -4 & -2t \\ 0 & 0 & 1 & 0 & -t \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Back substitution gives $w = 0$, $z = -t$, $-2y = 2z + 4w - 2t = 2(-t) + 4(0) - 2t = -4t$, or $y = 2t$, and $x = -y - 2z - w + t = -2t - 2(-t) - 0 + t = t$. Hence $(x,y,z,w) = (t, 2t, -t, 0)$, which can be written $\mathbf{x} = t\mathbf{s}$ with $\mathbf{s} = (1, 2, -1, 0)$ and $t = 1/(2\lambda)$. An alternative method is to write down the FOC's based on a direct computation:

$$\mathcal{L}'_x = 1 - \lambda(6x + 2y + 8z - 2w) = 0$$
$$\mathcal{L}'_y = 1 - \lambda(2x + 2y + 4z + 2w) = 0$$
$$\mathcal{L}'_z = 1 - \lambda(8x + 4y + 14z) = 0$$
$$\mathcal{L}'_w = 1 - \lambda(-2x + 2y + 8w) = 0$$

Also with this method, we see that $\lambda \neq 0$, and we can solve each equation for $1/\lambda$, and put the expressions for $1/\lambda$ in the four FOC's equal to each other. This gives the homogeneous linear system

$$\begin{aligned} 6x + 2y + 8z - 2w &= 2x + 2y + 4z + 2w \\ 2x + 2y + 4z + 2w &= 8x + 4y + 14z \\ 8x + 4y + 14z &= -2x + 2y + 8w \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4x + 4z - 2w &= 0 \\ -6x - 2y - 10z + 2w &= 0 \\ 10x + 2y + 14z - 8w &= 0 \end{aligned}$$

We can solve this system with Gaussian elimination: We get that z is a free variable, and back substitution gives $w = 0$, $y = -2z$, and $x = -z$. Hence $(x, y, z, w) = (-z, -2z, z, 0)$, and when we put this into the first FOC, we get $1/\lambda = -6z - 4z + 8z = -2z$. Hence $z = -1/(2\lambda)$, and $(x, y, z, w) = 1/(2\lambda) \cdot (1, 2, -1, 0) = t\mathbf{s}$, the same solution as we obtained using matrices above. With either method for solving the FOC's, we continue to compute $g(\mathbf{x})$:

$$g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \quad \Rightarrow \quad g(t\mathbf{s}) = (t\mathbf{s})^T A(t\mathbf{s}) = t\mathbf{s}^T A(t\mathbf{s}) = t^2(\mathbf{s}^T A\mathbf{s}) = t^2 g(\mathbf{s})$$

The constraint can therefore be written

$$g(t, 2t, -t, 0) = t^2 g(1, 2, -1, 0) = t^2 \cdot 2 = 18 \quad \Rightarrow \quad t^2 = 9 \quad \Rightarrow \quad t = \pm 3 = 3$$

since $\lambda > 0$. This means that $(x, y, z, w) = (3, 6, -3, 0)$, and since $t = 1/(2\lambda)$, we have $2\lambda = 1/3$, or $\lambda = 1/6$. Hence there is a unique candidate point $(x, y, z, w; \lambda) = (3, 6, -3, 0; 1/6)$ that satisfied the Kuhn-Tucker conditions. Since

$$h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 1/6) = x + y + z + w - \frac{1}{6}\left(g(x, y, z, w) - 18\right)$$

has Hessian $H(h) = -(1/6) \cdot H(g) = -(1/6) \cdot 2A$, and A is positive definite from (a), it follows that $H(h)$ is negative definite, and h is a concave function. Hence, it follows by the SOC that $f_{\max} = f(3, 6, -3, 0) = 6$ is the maximum value.

(e) Since A is a positive definite symmetric matrix, it has positive eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, and there is an orthogonal change of base $\mathbf{x} = P\mathbf{u}$ such that $g(\mathbf{x}) = \lambda_1 \cdot u_1^2 + \lambda_2 \cdot u_2^2 + \lambda_3 \cdot u_3^2 + \lambda_4 \cdot u_4^2$ in the new coordinates $\mathbf{u} = (u_1, u_2, u_3, u_4)$. Since D is given by the constraint $g(\mathbf{x}) \leq 18$, which can be written as

$$\lambda_1 \cdot u_1^2 + \lambda_2 \cdot u_2^2 + \lambda_3 \cdot u_3^2 + \lambda_4 \cdot u_4^2 \leq 18$$

it follows that $u_i^2 \leq 18/\lambda_i$ for $i = 1, 2, 3, 4$, or that

$$-\sqrt{18/\lambda_i} \leq u_i \leq \sqrt{18/\lambda_i}$$

Hence the set D is bounded in the new $\mathbf{u} = (u_1, u_2, u_3, u_4)$ coordinate system, and it is clearly a closed set. Therefore, D is a compact set.