Exam	Final exam in GRA 6035 Mathematics
Date	December 2nd, 2021 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ 4 & 2 & 2 & 0 \\ 1 & -2 & 8 & 5 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 2 \\ -3 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A, and find a base of the column space of A.
(b) **(6p)** Show that $\mathbf{v}$ is an eigenvector of A, and find the corresponding eigenvalue.
(c) **(6p)** Find the determinant of A.

Let S be a symmetric 3×3 matrix with eigenvalues $\lambda = 1$, $\lambda = 2$ and $\lambda = 4$.

(d) **(6p)** Show that S has an inverse matrix, and determine the definiteness of S^{-1}.

Question 2.

(a) **(6p)** Solve the differential equation $y'' + y' = 6e^{3t}$.
(b) **(6p)** Solve the differential equation $t(y' - y) = y$.
(c) **(6p)** Solve the difference equation $y_{t+2} + 3y_{t+1} - 4y_t = 5$.
(d) **(6p)** Solve the system of differential equations:

$$\mathbf{y}' = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix}$$

Question 3.

Let $g(x, y, z, w) = 3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 7z^2 + 4w^2$, and consider the Kuhn-Tucker problem given by

$$\max\ f(x, y, z) = x + y + z + w \text{ subject to } g(x, y, z, w) \leq 18$$

(a) **(6p)** Determine the definiteness of the quadratic form g.
(b) **(6p)** Write down the Kuhn-Tucker conditions of the problem in matrix form.
(c) **(6p)** Write down the non-degenerate constraint qualification in this problem, and find all admissible points where this condition does not hold (if there are any).
(d) **(6p)** Solve the Kuhn-Tucker problem.
(e) **Extra credit (6p)** Determine whether the set $D = \{(x, y, z, w) : g(x, y, z, w) \leq 18\}$ of admissible points is a compact set.