Solutions	Final exam in GRA 6035 Mathematics
Date	April 29th, 2022 at 1300 - 1600

Question 1.

(a) We find an echelon form of A to determine its rank and determinant. We start by adding -1 times the last row to the first row so that the first pivot is 1:

$$\begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 8 \\ 0 & -4 & -9 \end{pmatrix}$$

Then we add the last row to the middle row so that the second pivot is -1:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 8 \\ 0 & -4 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & -4 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$

We see that $\text{rk}\, A = 3$ since the matrix has three pivot positions, and the determinant is given by $\det A = 1 \cdot (-1) \cdot (-5) = 5$ since the elementary row operations we have used do not change the determinant. Alternatively, we could compute the determinant using cofactor expansion.

(b) We solve the linear system $(A - I)\mathbf{x} = \mathbf{0}$ to find a base of the nullspace:

$$A - I = \begin{pmatrix} 1 & -4 & -11 \\ -2 & 2 & 10 \\ 1 & -4 & -11 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & -11 \\ 0 & -6 & -12 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that z is a free variable (with x, y basic), hence $\dim \text{Null}(A - I) = 1$. We solve the linear system using back substitution, and find $y = -2z$ and $x = 3z$. The solutions are therefore given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

It follows that the vector $\mathbf{v}_1 = (3, -2, 1)$ forms a base of $\text{Null}(A - I) = E_1$.

(c) The eigenvalues of A are the solutions of the characteristic equation $|A - \lambda I| = 0$, and we compute the determinant on the left-hand side by cofactor expansion along the first column:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & -4 & -11 \\ -2 & 3-\lambda & 10 \\ 1 & -4 & -10-\lambda \end{vmatrix} \\ &= (2-\lambda) \cdot [(3-\lambda)(-10-\lambda) + 40] + 2(-4(-10-\lambda) - 44) + 1(-40 + 11(3-\lambda)) \\ &= (2-\lambda)(\lambda^2 + 7\lambda + 10) - 3\lambda - 15 = (2-\lambda)(\lambda+2)(\lambda+5) - 3(\lambda+5) \\ &= (\lambda+5)(4 - \lambda^2 - 3) = (\lambda+5)(1-\lambda^2) = 0 \end{aligned}$$

The solutions are therefore given by $\lambda + 5 = 0$, or $\lambda = -5$, or $1 - \lambda^2 = 0$, or $\lambda = \pm 1$. Hence the eigenvalues of A are $\lambda = 1, -1, -5$. Alternatively, we could have checked that the determinant $|A - \lambda I| = 0$ for $\lambda = -1, -5$, and refer to b) for $\lambda = 1$.

(d) Since there are three distinct eigenvalues of A, A is diagonalizable, and the eigenspaces E_{-1} and E_{-5} are one-dimensional. We find a base vector in each case: For $\lambda = -1$, an echelon form of $A + I$ is given by

$$\begin{pmatrix} 3 & -4 & -11 \\ -2 & 4 & 10 \\ 1 & -4 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ -2 & 4 & 10 \\ 1 & -4 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & -4 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

and for $\lambda = -5$, an echelon form of $A + 5I$ is given by

$$\begin{pmatrix} 7 & -4 & -11 \\ -2 & 8 & 10 \\ 1 & -4 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 20 & 19 \\ -2 & 8 & 10 \\ 1 & -4 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 20 & 19 \\ 0 & 48 & 48 \\ 0 & -24 & -24 \end{pmatrix} \to \begin{pmatrix} 1 & 20 & 19 \\ 0 & 48 & 48 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $\mathbf{v}_2 = (1, -2, 1)$ and $\mathbf{v}_3 = (1, 1, 1)$ are base vectors of E_{-1} and E_{-5}. We may therefore choose the matrix P with the bases of the eigenspaces as columns:

$$P = \begin{pmatrix} 3 & 1 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

We know from theory that this matrix satisfies $P^{-1}AP = D$, where D is the diagonal matrix with the eigenvalues $1, -1, -5$ on the diagonal.

Question 2.

(a) We use superposition to solve the **linear second order** differential equation $y'' - 3y' + 2y = 6e^{-t}$: To find the homogeneous solution y_h, we consider the characteristic equation $r^2 - 3r + 2 = 0$, with two distinct roots $r = 1$ and $r = 2$, and therefore

$$y_h = C_1 \, e^t + C_2 \, e^{2t}$$

To find a particular solution y_p, we consider the differential equation $y'' - 3y' + 2y = 6e^{-t}$. We try to find a solution of the form $y = Ae^{-t}$, which gives $y' = -Ae^{-t}$ and $y'' = Ae^{-t}$. When we substitute this into the differential equation, we get $(A + 3A + 2A)e^{-t} = 6e^{-t}$, or $6Ae^{-t} = 6e^{-t}$. We see that $A = 1$ is a solution, and the general solution is therefore given by

$$y = y_h + y_p = C_1 \, e^t + C_2 \, e^{2t} + e^{-t}$$

(b) The differential equation $ty' + y = 1$ can be written $y' + (1/t)y = 1/t$, and is **linear**. Since $\int 1/t \, \mathrm{d}t = \ln t + C$, the integrating factor is $u = e^{\ln t} = t$. Multiplication with the integrating factor gives

$$(t \cdot y)' = \frac{1}{t} \cdot t = 1 \quad \Rightarrow \quad t \cdot y = \int 1 \, \mathrm{d}t = t + C \quad \Rightarrow \quad y = \frac{t + C}{t}$$

(c) The differential equation $2ty' + y^2 = 1$ can be written $2ty' = 1 - y^2$, or $y' = (1 - y^2)/(2t)$. It is **separable** (but not linear), and we separate it and write it in the form

$$\frac{2}{1 - y^2} \, y' = \frac{1}{t} \quad \Rightarrow \quad \int \frac{2}{1 - y^2} \, \mathrm{d}y = \int \frac{1}{t} = \ln |t| + C$$

To solve the integral on the left-hand side, we use partial fractions, and find constants A, B such that

$$\frac{2}{(1 - y)(1 + y)} = \frac{A}{1 - y} + \frac{B}{1 + y} \quad \Rightarrow \quad 2 = A(1 + y) + B(1 - y) = (A + B) + (A - B)y$$

Comparing coefficients, we see that we need $A + B = 2$ and $A - B = 0$. This implies that $A = B$ and $2A = 2$, or $A = 1$. This gives

$$\int \frac{2}{1 - y^2} \, \mathrm{d}y = \int \frac{1}{1 - y} + \frac{1}{1 + y} \, \mathrm{d}y = -\ln |1 - y| + \ln |1 + y| + C = \ln \left| \frac{1 + y}{1 - y} \right| + C$$

When we substitute this into the equation above, we get

$$\ln \left| \frac{1 + y}{1 - y} \right| = \ln |t| + C \quad \Rightarrow \quad \left| \frac{1 + y}{1 - y} \right| = |t| \cdot e^C \quad \Rightarrow \quad \frac{1 + y}{1 - y} = Kt$$

where $K = \pm e^C$. Hence $1 + y = Kt(1 - y) = Kt - Kty$, or $y(1 + Kt) = Kt - 1$. The general solution on explicit form is therefore

$$y = \frac{Kt - 1}{Kt + 1}$$

(d) The equilibrium state $\mathbf{y}_e$ is the solutions of $\mathbf{y}' = \mathbf{0}$, or

$$\begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} \cdot \mathbf{y} + \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} \cdot \mathbf{y} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

We could solve this as a linear system using Gaussian elimination. But note that the coefficient matrix equals the matrix A in Question 1, and the vector on the right-hand side equals one of the eigenvectors $\mathbf{v}_1$ in E_1 that we found in Question 1b. This means that $A\mathbf{v}_1 = 1 \cdot \mathbf{v}_1 = \mathbf{v}_1$, and therefore $\mathbf{y}_e = \mathbf{v}_1 = (3, -2, 1)$. We found the eigenvalues of A in Question 1c, and since $\lambda = 1$ is a positive eigenvalue, we know that $\mathbf{y}_e = (3, -2, 1)$ is unstable.

Question 3.

(a) We write f on matrix form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x}$, where

$$A = \begin{pmatrix} -2 & -1 & -4 \\ -1 & 0 & 0 \\ -4 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 & 0 \end{pmatrix}$$

To find the stationary points, we solve the first order conditions $f'(\mathbf{x}) = 2A\mathbf{x} + B^T = \mathbf{0}$, or $A\mathbf{x} = -1/2 \cdot B^T$. This gives a linear system, and we solve it using Gaussian elimination:

$$\left(\begin{array}{ccc|c} -2 & -1 & -4 & 0 \\ -1 & 0 & 0 & -2 \\ -4 & 0 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 0 & -2 \\ -2 & -1 & -4 & 0 \\ -4 & 0 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 0 & -2 \\ 0 & -1 & -4 & 4 \\ 0 & 0 & -3 & 8 \end{array}\right)$$

Back substitution gives $-3z = 8$ or $z = -8/3$, $-y - 4(-8/3) = 4$, or $y = 20/3$, and that $-x = -2$ or $x = 2$. Hence $\mathbf{x}^* = (2, 20/3, -8/3)$ is the unique stationary point f. To classify it, notice that A is indefinite since $D_2 = -1$. This means that $H(f)(\mathbf{x}^*) = 2A$ is also indefinite, and $\mathbf{x}^* = (2, 20/3, -8/3)$ is a saddle point for f by the second derivative test. Alternatively, we could find the stationary point and classify it without using the matrix form of f.

(b) The Lagrange problem has Lagrangian $\mathcal{L} = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} - \lambda(\mathbf{x}^T D\mathbf{x} - 2)$, where D is the symmetric matrix of the quadratic form g, given by

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

The first order conditions (FOC) can therefore be written $\mathcal{L}'(\mathbf{x}) = 2A\mathbf{x} + B^T - \lambda(2D\mathbf{x}) = \mathbf{0}$, and the constraint (C) can be written $\mathbf{x}^T D\mathbf{x} = 2$. Together, the conditions FOC + C are the Lagrange conditions of the problem:

$$\text{FOC+C:} \quad 2A\mathbf{x} + B^T - \lambda(2D\mathbf{x}) = \mathbf{0},\ \mathbf{x}^T D\mathbf{x} = 2$$

Alternatively, we could write down the Lagrange conditions without using matrix forms.

(c) When $\lambda = 1$, the first order conditions are $2A\mathbf{x} + B^T - 2D\mathbf{x} = \mathbf{0}$, or $(A - D)\mathbf{x} = -1/2 \cdot B^T$. This is a linear system, and we solve it using Gaussian elimination (where the first step is to subtract the last row from the first to simplify computations):

$$\left(\begin{array}{ccc|c} -3 & -1 & -4 & 0 \\ -1 & -1 & -2 & -2 \\ -4 & -2 & -7 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ -1 & -1 & -2 & -2 \\ -4 & -2 & -7 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Back substitution gives $z = -2$, $2y + 5(-2) = 0$ or $y = 5$, and that $x + 5 + 3(-2) = 0$, or $x = 1$. We get the solution $(x, y, z; \lambda) = (1, 5, -2; 1)$ of the FOC. We see that the constraint is satisfied since

$$g(1, 5, -2) = (1)^2 + 5^2 + 4(-2)^2 + 4(5)(-2) = 2$$

at this point. We conclude that there is one candidate point $(x, y, z; \lambda) = (1, 5, -2; 1)$ with $\lambda = 1$ that satisfies the Lagrange conditions. Alternatively, we could find the candidate points without using matrix forms.

(d) We use the second order condition (SOC) to test the candidate point $(x, y, z; \lambda) = (1, 5, -2; 1)$, and therefore consider the function

$$h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 1) = 2A\mathbf{x} + B^T - 2D\mathbf{x} = 2(A - D)\mathbf{x} + B^T$$

We notice that h is a quadratic function with Hessian $H(h) = 2(A - D)$, and that $H(h)$ has the same definiteness as

$$A - D = \begin{pmatrix} -3 & -1 & -4 \\ -1 & -1 & -2 \\ -4 & -2 & -7 \end{pmatrix}$$

We compute the principal minors of $A - D$: We have $D_1 = -3$, $D_2 = 3 - 1 = 2$, and that $D_3 = -3(7 - 4) + 1(7 - 8) - 4(2 - 4) = -2$. We conclude that $A - D$, and therefore

$H(h) = 2(A - D)$, is negative definite, and it follows that h is a concave function. By the SOC, it follows that $(x, y, z) = (1, 5, -2)$ is a maximizer in the Lagrange problem, and that $f_{\max} = f(1, 5, -2) = 12$ is the maximum value. Alternatively, we could apply the SOC without using matrix forms.

(e) To find the linear change of variables, we find the eigenvalues and eigenvectors of the symmetric matrix D of the quadratic form g, given by

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

The characteristic equation is given by $(1 - \lambda)(\lambda^2 - 5\lambda) = 0$, and the eigenvalues are therefore $\lambda = 1, 0, 5$. Next, we find a base for each eigenspace, given by the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

respectively. We know that D is orthogonal diagonalizable since D is symmetric, and to find an orthonormal set of base vectors in each case, we divide each vector by its length (since all eigenspace have dimension one). This gives orthonormal bases

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{w}_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

respectively. It follows that the orthogonal matrix P with these vectors as columns satsify

$$P^T D P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

and this means that $g(\mathbf{x}) = 2$ can be written $u_1^2 + 5u_3^2 = 2$ when we use the linear change of base given by

$$\mathbf{x} = P\mathbf{u} \quad \Leftrightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

This means that the set of admissible points is an elliptical cylinder.