Exam	Final exam in GRA 6035 Mathematics
Date	November 30th 2022 at 1400 - 1700

You must give reasons for your answers.

Question 1.
We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A.
(b) **(6p)** Find a base of the nullspace of A.
(c) **(6p)** Determine the eigenvalue λ such that $\mathbf{v}$ is in the eigenspace E_λ.
(d) **(6p)** Find a base of the space U of vectors that are orthogonal to all vectors in $\text{Null}(A)$.

Question 2.
Let q be the quadratic form given by $q(x, y, z) = 3x^2 + 4xy + 2xz + 4y^2 + 2yz + z^2$, and let p be the function given by $p(x, y, z) = u \cdot e^u$ with $u = u(x, y, z) = 1 - q(x, y, z)$.

(a) **(6p)** Determine the definiteness of the quadratic form q.
(b) **(6p)** Solve the unconstrained problem $\max / \min \; p(x, y, z)$.

Consider the Lagrange problem $\max / \min f(x, y, z) = x + y + z$ when $q(x, y, z) = 4$, where q is the quadratic form given above.

(c) **(6p)** Assume that $(x^*, y^*, z^*; \lambda^*)$ is an admissible point that satisfies the first order conditions. Determine which of the following statements are true, and give reasons for your answers:
  (A) If $\lambda^* > 0$, then (x^*, y^*, z^*) is a minimum point
  (B) If $\lambda^* > 0$, then (x^*, y^*, z^*) is a maximum point
  (C) If $\lambda^* < 0$, then (x^*, y^*, z^*) is a minimum point
  (D) If $\lambda^* < 0$, then (x^*, y^*, z^*) is a maximum point
(d) **(6p)** Solve the Lagrange problem.

Question 3.

(a) **(6p)** Solve the differential equation: $4y'' + 4y' - 3y = 8 + 8t - 3t^2$
(b) **(6p)** Solve the system of difference equations: $u_{t+1} = 0.7u_t + 0.8v_t, \; v_{t+1} = 0.4u_t + 0.3v_t$
(c) **(6p)** Solve the initial value problem: $2ty^2 - 4y + (2t^2y - 4t)y' = 0, \; y(1) = 5$
(d) **(6p)** Solve the differential equation: $t^2y'' + 4ty' + 2y = 6$

Hint: To find the homogeneous solution in (d), determine all values of r such that $y = t^r$ is a solution of the homogeneous differential equation.