Exam	Final exam in GRA 6035 Mathematics
Date	January 10th 2024 at 0900 - 1400

You must give reasons for your answers.

Question 1.

(a) **(3p)** Find the general solution of the differential equation $y'' - 2y' = 0$.

(b) **(3p)** Determine whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 4 \end{pmatrix}$$

(c) **(3p)** Show that $(2, 2, 2)$ is a stationary point of f and classify it as a local maximum point, local minimum point or saddle point of f when $f(x, y, z) = 3xy + 3xz + 3yz - x^3 - y^3 - z^3$.

(d) **(3p)** Determine whether the system $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ of linear differential equations has a stable equilibrium state when

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

Question 2.

We consider the matrix A given by

$$A = \begin{pmatrix} 2 & 7 & 3 \\ 3 & 11 & 5 \\ 1 & -4 & 0 \end{pmatrix}$$

(a) **(6p)** Compute the determinant and the rank of A.

(b) **(6p)** Solve the linear system $A\mathbf{x} = \mathbf{x}$. How many solutions are there?

(c) **(6p)** Find the characteristic equation of A, and determine whether A is diagonalizable.

Question 3.

(a) **(6p)** Find the general solution of the difference equation $y_{t+2} + y_{t+1} - 6y_t = 3 - 4t$.

(b) **(6p)** Find the general solution of the differential equation $t + y' = y$.

(c) **(6p)** Find the general solution of the system $\mathbf{y}_{t+1} = A\mathbf{y}_t + \mathbf{b}$ of difference equations, where

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(d) **(6p)** Find the particular solution of the differential equation that satisfy the initial condition:

$$t - 3y + (8y - 3t) \cdot y' = 0, \quad y(1) = 0$$

Question 4.

Let $f(x, y, z, w) = 2xz + 2xw + 2yz + 2yw$ and consider the Lagrange problem

$$\max / \min f(x, y, z, w) \text{ when } x^2 + 2y^2 + 2z^2 + 6w^2 = 48$$

(a) **(6p)** Find the candidate points $(x, y, z, w; \lambda)$ in the Lagrange problem with $\lambda = 1$.

(b) **(6p)** Find the symmetric matrix of the quadratic form f, and determine its definiteness.

(c) **(6p)** Show that the Lagrange problem has a maximum, and find the maximum value.

(d) **(6p)** Estimate the new maximum value if we change the constraint to $x^2+2y^2+z^2+6w^2 = 48$.

(e) **(6p)** Show that the (original) Lagrange problem has a minimum, and find the minimum value.