Solutions	Midterm exam in GRA 6035 Mathematics
Date	January 6th, 2017 at 0900 - 1000

Correct answers: C-D-C-B-D-D-A-A

Question 1.

Since $\operatorname{rk} A = 5$ and the linear system has 5 equations, the linear system is consistent. It has $6 - 5 = 1$ degrees of freedom. The correct answer is alternative **C**.

Question 2.

We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and compute its determinant

$$\begin{vmatrix} 4 & s & 3 \\ -4 & 3 & 1 \\ 2s & 0 & 1 \end{vmatrix} = 2s(s-9) + 1(12+4s) = 2s^2 - 14s + 12 = 2(s^2 - 7s + 6) = 2(s-6)(s-1)$$

This shows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent when $s = 1, 6$, and linearly independent otherwise. The correct answer is alternative **D**.

Question 3.

We use elementary row operations to find an echelon form of the coefficient matrix A:

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 3 & 1 & 4 & -1 \\ 1 & s & s+1 & 9 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -5 & -7 \\ 0 & s-2 & s-2 & 7 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -5 & -7 \\ 0 & 0 & 0 & * \end{pmatrix}$$

where $* = 7 - 7(s+2)/5 = 7(7-s)/5$. This means that there are two free variables if $s = 7$, and one free variable if $s \neq 7$. The correct answer is alternative **C**.

Question 4.

We have that $\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} A = 4$ and that $\lambda_1\lambda_2\lambda_3 = \det(A) = -16$. We can also compute $\lambda_1 = -2$, $\lambda_2 = 2$ and $\lambda_3 = 4$ explicitly to see this. The correct answer is alternative **B**.

Question 5.

The eigenvalues are $\lambda = 1$ and $\lambda = s$. When $s \neq 1$, the eigenspace $(A - I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ has one free variable while the multiplicity of $\lambda = 1$ is two. When $s = 1$, the eigenspace $(A - I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ has one free variables while the multiplicity of $\lambda = 1$ is three. In both cases, A is not diagonalizable. The correct answer is alternative **D**.

Question 6.

Eigenvectors for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.29 & 0.29 \\ 0.29 & -0.29 \end{pmatrix}$$

Therefore, we see that $x = 1$ and $y = 1$ gives one eigenvector, and all others are multiple of this one. Multiplication by $1/2$ gives the state vector with $x = 1/2$ and $y = 1/2$. The correct answer is alternative **D**.

Question 7.

The symmetric matrix of the quadratic form $f(x,y,z) = 5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2$ is given by

$$A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

The leading principal minors are $D_1 = 5$, $D_2 = 25 - 16 = 9$ and $D_3 = 8(9) + 2(-18) - 2(18) = 0$, and we compute all principal minors $\Delta_1 = 5, 5, 8$, $\Delta_2 = 9, 36, 36$ and $\Delta_3 = 0$. Since all principal minors $\Delta_i \geq 0$, f is positive semidefinite (but not positive definite since $D_3 = 0$). The correct answer is alternative **A**.

Question 8.

The function $f(x,y) = x/y + y/x$ has Hessian matrix

$$H(f) = \begin{pmatrix} 2y/x^3 & -1/y^2 - 1/x^2 \\ -1/y^2 - 1/x^2 & 2x/y^3 \end{pmatrix}$$

Hence $D_1 = 2y/x^3 > 0$ and this means that f may be convex but not concave. D_2 is given by

$$D_2 = \frac{4}{x^2y^2} - \left(\frac{-x^2 - y^2}{x^2y^2}\right)^2 = \frac{-x^4 + 2x^2y^2 - y^4}{x^4y^4} = -\frac{(x^2 - y^2)^2}{x^4y^4}$$

Since $D_2 \leq 0$ can take negative values, f is not convex. The correct answer is alternative **A**.