Solutions	Midterm exam in GRA 6035 Mathematics
Date	October 08th, 2021 at 1500 - 1600

Correct answers: D-C-B-A A-A-B-C

Question 1.
We find the pivot positions in A, given by the Gaussian process

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 4 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -1 & -1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence there are two degrees of freedom. The correct answer is alternative **D**.

Question 2.
We find the pivot positions in A, given by the Gaussian process

$$\left(\begin{array}{cccc|} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 5 & 4 & 6 & 1 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a base of $\text{Col}(A)$. The correct answer is alternative **C**.

Question 3.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 5-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the first row, which gives

$$(3-\lambda) \cdot \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 10\lambda + 21)$$

Since $\lambda^2 - 10\lambda + 21 = 0$ has roots $\lambda = 3$ and $\lambda = 7$, there is one eigenvalue $\lambda = 3$ of multiplicity two, and another with multiplicity one. The correct answer is alternative **B**.

Question 4.
We compute the 2-minors of A:

$$M_{12,12} = t, \quad M_{12,23} = t, \quad M_{12,13} = 2t$$

Hence $\text{rk}\, A < 2$ if and only if $t = 0$, and in this case $\text{rk}\, A = 1$. The correct answer is alternative **A**.

Question 5.
Since the eigenvalues of A are $\lambda = 3, s, 1$, it has three distinct eigenvalues when $s \neq 1, 3$. In case $s = 1$, the eigenvalue $\lambda = 1$ has multiplicity two, and the eigenspace is the null space of

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there is one free variable, A is not diagonalizable for $s = 1$. In case $s = 3$, the eigenvalue $\lambda = 3$ has multiplicity two, and the eigenspace is the null space of

$$\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

Since there are two free variables, A is diagonalizable for $s = 3$. The correct answer is alternative **A**.

Question 6.
The symmetric matrix of the quadratic form $f(x, y, z, w) = 3x^2+2xy+8xz-2xw+y^2+4yz+2yw+6z^2$ is given by

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 6 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

Since the principal minor $\Delta_2 = M_{24,24} = 0 - 1 = -1 < 0$, it follows that A is indefinite. The correct answer is alternative **A**.

Question 7.
The function $f(x, y, z) = x^4+y^4+z^4-4xyz$ has first order partial derivatives and first order conditions given by

$$f'_x = 4x^3 - 4yz = 0, \quad f'_y = 4y^3 - 4xz = 0, \quad f'_z = 4z^3 - 4xy = 0$$

Since $(1, 1, 1)$ satisfy these conditions, it is a stationary point, and we have that the Hessian matrix at $(1, 1, 1)$ is given by

$$H(f) = \begin{pmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{pmatrix} \quad \Rightarrow \quad H(f)(1, 1, 1) = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{pmatrix}$$

Since this matrix has $D_1 = 12$, $D_2 = 128$, and $D_3 = 1024$, it is positive definite, and $(1, 1, 1)$ is a local minimum point for f. The correct answer is alternative **B**.

Question 8.
Since $\dim E_0 = \dim \text{Null}(A) = 1$, we have that $A\mathbf{x} = \mathbf{0}$ has one degree of freedom, and therefore the rank of A is $4 - 1 = 3$. The correct answer is alternative **C**.