Solutions	Midterm exam in GRA 6035 Mathematics
Date	October 14th, 2022 at 0900 - 1000

Question 1.
We have $\dim \text{Null}(A) = n - \text{rk}(A) = 4 - 2 = 2$ since A has an echelon form with two pivots:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Question 2.
We can write $\mathbf{v}_4 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ since Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & 3 & 1 & 4 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & 3 & 1 & 4 \\ 0 & -3 & -7 & -10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & -6 & -6 \end{array}\right)$$

and the unique solution is $(-1, 1, 1)$ by back substitution.

Question 3.
The rank of A is maximal for $t \neq 1, t \neq 2$ since $\text{rk}(A) < 3$ if and only if

$$|A| = \begin{vmatrix} 1 & 1 & t \\ t & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} = 1(5) - 1(3t - 3) + t(4t - 9) = 4t^2 - 12t + 8 = 4(t - 1)(t - 2) = 0$$

Question 4.
The equilibrium state is $\mathbf{v} = (1/5, 4/5)$ since the Markov chain is regular, and the eigenvector $(1, 4)$ is a base of the eigenspace

$$E_1 = \text{Null} \begin{pmatrix} -0.28 & 0.07 \\ 0.28 & -0.07 \end{pmatrix} = \text{Null} \begin{pmatrix} -4 & 1 \\ 0 & 0 \end{pmatrix}$$

with $1/5 \cdot (1, 4) = (1/5, 4/5)$ as the unique state vector in E_1.

Question 5.
The quadratic form q is negative semidefinite by the RRC since its symmetric matrix A has $\text{rk}\, A = 1$ and $D_1 = -1 < 0$:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \\ 1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 6.
We have that $\mathbf{v}$ is in E_2 since $A \cdot \mathbf{v} = 2\mathbf{v}$, and $\dim E_2 = \dim \text{Null}(A - 2I) = n - \text{rk}(A - 2I) = 4 - 3 = 1$ since $A - 2I$ has an echelon form form with three pivots:

$$A - 2I = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 7.
The scalar $a = 1/2$ since $\mathbf{v} - \text{proj}_{\mathbf{w}}(\mathbf{v})$ is orthogonal to $\mathbf{w}$ and the projection is given by

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \cdot \mathbf{w} = \frac{26}{52} \cdot (1, 1, 1, 7) = \frac{1}{2} \cdot w$$

Question 8.
The point $(-1, 5, -1, 0)$ is the unique solution of the linear system with $w = 0$: The 3×4 linear system has one degree of freedom since $M_{123,124} \neq 0$, hence the solutions are the points on the line

$$(x, y, z, w) = (1, 1, 3, 4) + t(-1, 2, -2, -2) = (1 - t, 1 + 2t, 3 - 2t, 4 - 2t)$$

Moreover, $w = 0$ gives $t = 2$, and the point is $(-1, 5, -1, 0)$.