BI NORWEGIAN BUSINESS SCHOOL

Solutions:	**GRA 60353 Mathematics**		
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Permitted examination support material:	A bilingual dictionary and BI-approved calculator TEXAS INSTRUMENTS BA II Plus		
Answer sheets:	Squares		
	Counts 80% of GRA 6035		The subquestions have equal weight
Mock exam			Responsible department: Economics

Question 1.

(a) The partial derivatives of $f(x,y,z,w) = x^2 - y^2 + y^3 + yz + z^2 + w^2$ are given by

$$f'_x = 2x, \quad f'_y = -2y + 3y^2 + z, \quad f'_z = y + 2z, \quad f'_w = 2w$$

and its Hessian matrix is given by

$$H(f)(x,y,z,w) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 6y-2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(b) The stationary points of f are given by

$$f'_x = 2x = 0, \quad f'_y = -2y + 3y^2 + z = 0, \quad f'_z = y + 2z = 0, \quad f'_w = 2w = 0$$

This gives $x = w = 0$, $y = -2z$ and $-2y + 3y^2 + z = 4z + 12z^2 + z = 0$. The last equation is $5z + 12z^2 = z(5 + 12z) = 0$, with solutions $z = 0$ and $z = -5/12$. This gives two stationary points $(x,y,z,w) = (0,0,0,0)$ and $(x,y,z,w) = (0, 10/12, -5/12, 0)$. The Hessian matrix at $(0,0,0,0)$ has $D_1 = 2$ and $D_2 = -4$, so this matrix is indefinite. The Hessian matrix at $(0, 10/12, -5/12, 0)$ has $D_1 = 2$, $D_2 = 2 \cdot 3 = 6$, $D_3 = 2 \cdot (2(6y-2) - 1) = 2 \cdot 5 = 10$, and $D_4 = 2D_3 = 20$, so this matrix is positive definite. It follows that $(0,0,0,0)$ is a saddle point and that $(0, 10/12, -5/12, 0)$ is a local minimum point.

(c) If the function f was convex or concave, the Hessian matrix $H(f)(x,y,z,w)$ would be either positive semidefinite at all points (x,y,z,w), or negative semidefinite at all points (x,y,z,w). This is not the case, since the Hessian is indefinite at $(0,0,0,0)$. It follows that f is not convex and not concave.

Question 2.

(a) We compute the determinant of A by cofactor expansion along the first column:

$$\det(A) = \begin{vmatrix} a & 1 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = a\left(a \cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\right) - 1\left(1 \cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\right) = (a^2 - 1) \cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

We therefore have $\text{rk}(A) \leq 3$ for all a. To find the rank of A, we use row operations to obtain an echelon form (and start by interchanging the first two rows):

$$\begin{pmatrix} a & 1 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1-a^2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are two pivot positions when $a^2 = 1$ and three otherwise, we have that

$$\text{rk}(A) = \begin{cases} 3, & a \neq \pm 1 \\ 2, & a = \pm 1 \end{cases}$$

(b) The symmetric matrix A has leading principal minors D_i given by

$$D_1 = a, \quad D_2 = a^2 - 1, \quad D_3 = a^2 - 1, \quad D_4 = |A| = 0$$

Since $D_4 = 0$, it is necessary to compute all principal minor to find out when A is positive semidefinite, and we find the following principal minors:

$$\begin{aligned} \Delta_1 &= a, a, 1, 1 \\ \Delta_2 &= a^2 - 1, a, a, a, a, 0 \\ \Delta_3 &= a^2 - 1, a^2 - 1, 0, 0 \\ \Delta_4 &= 0 \end{aligned}$$

All principal minors $\Delta_i \geq 0$ when $a \geq 0$ and $a^2 - 1 \geq 0$, so A is positive semidefinite for $a \geq 1$ (and indefinite when $a < 1$).

(c) The characteristic polynomial of A (the left side of the characteristic equation) is given by

$$\begin{vmatrix} a-\lambda & 1 & 0 & 0 \\ 1 & a-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & -1 & 1-\lambda \end{vmatrix} = (a-\lambda)\left((a-\lambda) \cdot \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}\right) - 1\left(1 \cdot \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}\right)$$

Hence the characteristic equation is given by

$$\left((a-\lambda)^2 - 1\right) \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \left((a-\lambda)^2 - 1\right)(\lambda^2 - 2\lambda) = 0$$

which can be expressed as $(a-\lambda)^2 - 1 = 0$ or $\lambda^2 - 2\lambda = 0$. The eigenvalues are therefore $\lambda = a+1$, $\lambda = a-1$, $\lambda = 0$ and $\lambda = 2$.

Question 3.

(a) The differential equation $y'' - 7y' + 10y = 4e^t - 5$ is second order linear, and it has solution $y = y_h + y_p$. The homogeneous equation $y'' - 7y' + 10y = 0$ has characteristic equation $r^2 - 7r + 10 = 0$, and distinct roots $r = 2$ and $r = 5$. Therefore $y_h = C_1 e^{2t} + C_2 e^{5t}$. To find a particular solution y_p, we consider the right hand side $f(t) = 4e^t - 5$ and its derivatives $f' = 4e^t$ and $f'' = 4e^t$. We guess that there is a solution of the form $y = Ae^t + B$. Inserting this guess in the differential equation, we obtain

$$Ae^t - 7(Ae^t) + 10(Ae^t + B) = 4e^t - 5$$

or $4Ae^t + 10B = 4e^t - 5$. We see that $A = 1$ and $B = -1/2$ is a solution, so $y_p = e^t - 1/2$ and the general solution is

$$y = y_h + y_p = C_1 e^{2t} + C_2 e^{5t} + e^t - 1/2$$

(b) The differential equation $ty' + (2-t)y = e^{2t}$ is first order linear since it can be written in standard form as

$$y' + \frac{2-t}{t}y = t^{-1}e^{2t}$$

It can be solved using integrating factor, and

$$\int \frac{2-t}{t}\,dt = \int (2/t - 1)\,dt = 2\ln t - t + \mathcal{C}$$

so the integrating factor is $u = e^{2\ln t - t} = t^2e^{-t}$. After multiplying with the integrating factor, we get

$$(yu)' = t^{-1}e^{2t}u = te^t \quad \Rightarrow \quad y = \frac{1}{u}\int te^t\,dt = \frac{te^t - e^t + \mathcal{C}}{t^2e^{-t}} = \frac{t-1}{t^2}e^{2t} + \frac{\mathcal{C}}{t^2}\,e^t$$

(c) The differential equation $3y^2te^{-t}y' + (y^3-1)e^{-t} = te^{-t}y^3$ can be written as $py' + q = 0$ with

$$p = 3y^2te^{-t}, \quad q = (y^3-1)e^{-t} - te^{-t}y^3$$

We try to find a function $h = h(y,t)$ such that $h'_y = p$ and $h'_t = q$. From the first condition, we get

$$h = y^3te^{-t} + \phi(t)$$

and using this expression for h, the second condition becomes $h'_t = q$, where

$$h'_t = y^3(1\cdot e^{-t} + te^{-t}(-1)) + \phi'(t) = y^3e^{-t} - y^3te^{-t} + \phi'(t)$$
$$q = (y^3-1)e^{-t} - te^{-t}y^3 = y^3e^{-t} - e^{-t} - y^3te^{-t}$$

Hence $h'_t = q$ holds if $\phi'(t) = -e^{-t}$. We may therefore choose $\phi(t) = e^{-t}$, and we find a function $h = y^3te^{-t} + e^{-t}$ that satisfies $h'_y = p$ and $h'_t = q$. This means that the differential equation is exact, with solution

$$h = y^3te^{-t} + e^{-t} = \mathcal{C} \quad \Rightarrow \quad y = \sqrt[3]{\frac{\mathcal{C}e^t - 1}{t}}$$

Question 4.

(a) We write the Kuhn-Tucker problem in standard form as

$$\max\ -f(x,y,z,w) = -x^2 - y^2 - z^2 - w^2 \text{ subject to } \begin{cases} xy + 1 \le 0 \\ 2zw + 8 \le 0 \end{cases}$$

and we form the Lagrangian

$$\mathcal{L} = -x^2 - y^2 - z^2 - w^2 - \lambda_1(xy+1) - \lambda_2(2zw+8)$$

The first order conditions (FOC) are

$$\mathcal{L}'_x = -2x - \lambda_1 y = 0$$
$$\mathcal{L}'_y = -2y - \lambda_1 x = 0$$
$$\mathcal{L}'_z = -2z - \lambda_2\cdot 2w = 0$$
$$\mathcal{L}'_w = -2w - \lambda_2\cdot 2z = 0$$

the constraints (C) are given by $xy+1 \le 0$ and $2zw+8 \le 0$, and the complementary slackness conditions (CSC) are given by

$$\lambda_1 \ge 0 \quad \text{and } \lambda_1(xy+1) = 0$$
$$\lambda_2 \ge 0 \quad \text{and } \lambda_2(2zw+8) = 0$$

When $(x,y,z,w) = (1,-1,2,-2)$, the FOC's give $-2+\lambda_1 = 0$ and $-4+4\lambda_2 = 0$, or $\lambda_1 = 2$ and $\lambda_2 = 1$. Since $xy = -1$ and $zw = -4$, the C's are satisfied and binding, and the CSC's are satisfied. So $(x,y,z,w;\lambda_1,\lambda_2) = (1,-1,2,-2;2,1)$ is a solution of the K-T conditions.

(b) We prove that $(x, y, z, w) = (1, -1, 2, -2)$ is max for $-f$ (and a min for f) using the SOC: If $h(x, y, z, w) = \mathcal{L}(x, y, z, w; 2, 1) = -x^2 - y^2 - z^2 - w^2 - 2(xy+1) - (2zw+8)$ is a concave funtion in (x, y, z, w), then $(x, y, z, w) = (1, -1, 2, -2)$ is a maximum point for $-f$. The Hessian matrix of h is

$$H(h) = \begin{pmatrix} -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

This is a symmetric matrix with leading principal minors $D_1 = -2$, $D_2 = 0$, $D_3 = 0$ and $D_4 = 0$. We compute the principal minors of order one and two:

$$\Delta_1 = -2, -2, -2, -2$$
$$\Delta_2 = 0, 4, 4, 4, 4, 0$$

The Hessian matrix is clearly of rank two, so all principal minors of order three and four are zero. This implies that h is a concave function, and therefore $(x, y, z, w) = (1, -1, 2, -2)$ solves the KT problem. The minimum value of f is $f(1, -1, 2, -2) = 1 + 1 + 4 + 4 = 10$.

(c) Let us consider the KT problem

$$\max\ -f(x, y, z, w) = -x^2 - y^2 - z^2 - w^2 \text{ subject to } \begin{cases} xy + 1 \le 0 \\ 2zw + c \le 0 \end{cases}$$

with Lagrangian $\mathcal{L} = -x^2 - y^2 - z^2 - w^2 - \lambda_1(xy + 1) - \lambda_2(2zw + c)$. When $c = 8$, we have found the maximum value $-f = -(1 + 1 + 4 + 4) = -10$. When we change c to $c = 7.9$, it follows from the Envelope theorem that the change in maximum value is estimated by

$$\Delta c \cdot \mathcal{L}'_c(x^*(c), y^*(c), z^*(c), w^*(c); \lambda_1^*(c), \lambda_2^*(c)) = -0.1 \cdot (-\lambda_2^*(8)) = 0.1$$

since $\lambda_2^*(8) = 1$. The new maximal value for $-f$ is approximately $-10 + 0.1 = -9.9$, and the new minimum value for f is approximately $f = 9.9$.

(d) Consider the first order conditions (FOC) given by

$$\mathcal{L}'_x = -2x - \lambda_1 y = 0$$
$$\mathcal{L}'_y = -2y - \lambda_1 x = 0$$
$$\mathcal{L}'_z = -2z - \lambda_2 2w = 0$$
$$\mathcal{L}'_w = -2w - \lambda_2 2z = 0$$

If $\lambda_1 = 0$, then $x = y = 0$, and this does not satisfy the first constraint. If $\lambda_2 = 0$, then $z = w = 0$, and this does not satisfy the second constraint. Therefore $\lambda_1, \lambda_2 > 0$, and $xy = -1$ and $zw = -4$ by the CSC's. In particular, $x, y, z, w \neq 0$. The first two conditions give $y = -2x/\lambda_1$ and $x = -2y/\lambda_1$. When we combine these condtions, we get

$$x = -2y/\lambda_1 = (2/\lambda_1)^2 x \quad \Rightarrow \quad \lambda_1 = 2$$

since $x \neq 0$. This implies that $x = -y$ and since $xy = -1$, we must have $x^2 = 1$. We get two solutions $(x, y) = (1, -1)$ or $(x, y) = (-1, 1)$ with $\lambda_1 = 2$. The last two FOC's give $w = -z/\lambda_2$ and $z = -w/\lambda_2$. When we combine these condtions, we get

$$z = -w/\lambda_2 = (1/\lambda_1)^2 z \quad \Rightarrow \quad \lambda_2 = 1$$

since $z \neq 0$. This implies that $z = -w$ and since $zw = -4$, we must have $z^2 = 4$. We get two solutions $(z, w) = (2, -2)$ or $(x, y) = (-2, 2)$ with $\lambda_2 = 1$. All four candidates satisfy C + CSC as well as FOC, so the points

$$(x, y, z, w; \lambda_1, \lambda_2) = (1, -1, 2, -2; 2, 1)$$
$$(x, y, z, w; \lambda_1, \lambda_2) = (-1, 1, 2, -2; 2, 1)$$
$$(x, y, z, w; \lambda_1, \lambda_2) = (1, -1, -2, 2; 2, 1)$$
$$(x, y, z, w; \lambda_1, \lambda_2) = (-1, 1, -2, 2; 2, 1)$$

are the solutions of the KT conditions.