Solutions

Mock exam problems in Differential equations.

Nov. 2017

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1. a) $y'' - 2y' - 3y = 4e^t$

second order linear diff. eqn.

$$y = y_h + y_p = \underline{\underline{C_1 e^{3t} + C_2 e^{-t} - e^t}}$$

$\underline{y_h}$: $y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$ char. eqn.

$$r = \frac{2 \pm \sqrt{4 - 4(-3)}}{2}$$

$$= \frac{2 \pm 4}{2} = 3, -1 \longrightarrow y_h = \underline{C_1 \cdot e^{3t} + C_2 e^{-t}}$$

$\underline{y_p}$: $y'' - 2y' - 3y = 4e^t \rightarrow$ $f = 4e^t$, $f' = 4e^t$, $f'' = 4e^t$

$y = A \cdot e^t$
$y' = Ae^t$
$y'' = Ae^t$

$Ae^t - 2(Ae^t) - 3(Ae^t) = 4e^t$

$e^t \cdot (A - 2A - 3A) = 4e^t$

$-4A = 4$

$A = -1 \longrightarrow y = \underline{-e^t}$

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b) $2ty - 1 + t^2 y' = 0$ exact?

$h'_t + h'_y \cdot y' = 0$

① $h'_t = 2ty - 1$

② $h'_y = t^2$

① $\Rightarrow h = t^2 y - t + \alpha(y)$

②: $h'_y = (t^2 y - t + \alpha(y))'$

$= t^2 - 0 + \alpha'(y)$

$= t^2 + \alpha'(y)$

$\Downarrow$

② $t^2 + \alpha'(y) = t^2$

$\alpha'(y) = 0$

can choose $\alpha(y) = 0$

$\Downarrow$

$h = t^2 y - t$

$\Leftarrow$

Conclusion:

The diff. eqn. is exact and can be written

$h'_t + h'_y \cdot y' = 0$

with $h = t^2 y - t$

$\Downarrow$

$h = t^2 y - t = C$

$t^2 y = C + t$

$y = \dfrac{C + t}{t^2}$

$$y = \frac{C}{t^2} + \frac{1}{t}$$

c) $2t y \cdot y' = 1$ Separable

$$2y \cdot y' = \frac{1}{t}$$

$$\int 2y\, y'\, dt = \int \frac{1}{t}\, dt$$

$$\int 2y\, dy = \int \frac{1}{t}\, dt$$

$$y^2 = \ln|t| + C$$

$$\underline{\underline{y = \pm\sqrt{\ln|t| + C}}}$$

2. a) $y' = 4e^y - 2$ $\underline{\text{autonomous}}$ $F(y) = 4e^y - 2$

Eq. states: $F(y) = 4e^y - 2 = 0$

$$4e^y = 2$$

$$e^y = 2/4 = 1/2$$

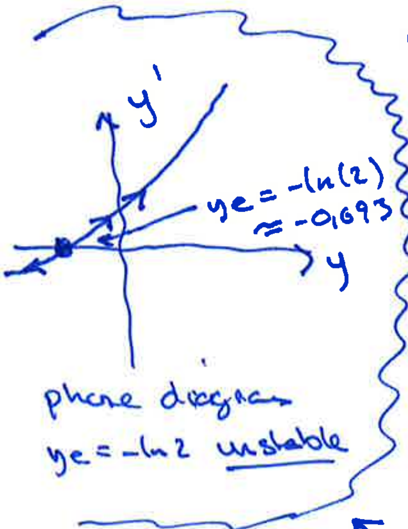
$$y_e = \ln(1/2) = \underline{\underline{-\ln(2)}}$$

Stability:

$$F'(y) = 4e^y$$

$$F'(y_e) = 4 \cdot e^{\ln(1/2)} = 4 \cdot \tfrac{1}{2} = 2 > 0$$

By the Stability Thm, $y_e = -\ln(2)$ is $\underline{\underline{\text{unstable}}}$



Since there are no stable eq. states, there cannot be globally asympt. stable eq. states either.

Alternative argument for the fact that $y_e = -\ln(2)$ is unstable: phase diagram.

b) $y' = y^2 - 3y + 2$

autonomous
$F(y) = y^2 - 3y + 2$

Eq. states: $F(y) = y^2 - 3y + 2 = 0$

$$y = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2} = \frac{3 \pm 1}{2} = 2, 1$$

$\underline{\underline{y_e = 1}}$, $\underline{\underline{y_e = 2}}$ eq. states.

Stability: $F'(y) = 2y - 3$

$F'(1) = 2 \cdot 1 - 3 = -1 < 0$

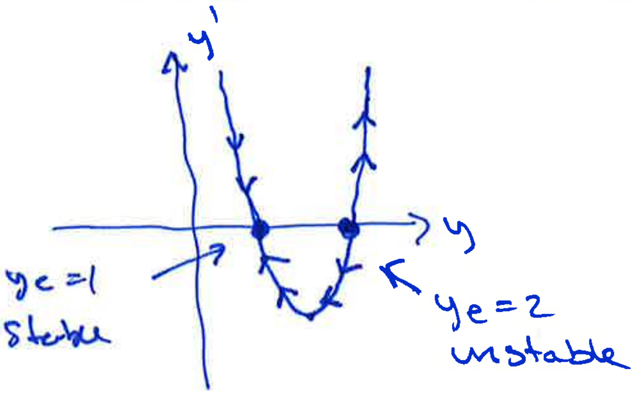
$F'(2) = 2 \cdot 2 - 3 = 1 > 0$

Stability Thm:

$y_e = 1$ stable

$y_e = 2$ unstable

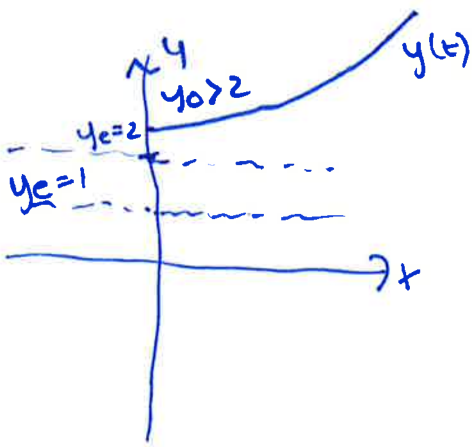
To check if $y_e = 1$ is globally asymp. stable:



phase diagram

We see from the phase diagram that

$y_0 > 2 \Rightarrow y' > 0$, y increasing and the curve $y(t)$ will not move towards $y_e = 1$ as t increases.



Therefore, $y_e = 1$ is stable but not globally asympt. stable.

3. $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$

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a) $A = \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix}$: We need to find eigenvalues and eigenvectors for A, and check if A is diagonalizable.

Eigenvalues: $\begin{vmatrix} 4-\lambda & -2 \\ -5 & 1-\lambda \end{vmatrix} = 0$

$(4-\lambda)(1-\lambda) - 10 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$\lambda = \frac{5 \pm \sqrt{25 - 4\cdot(-6)}}{2}$

$= \frac{5 \pm 7}{2} = 6, -1$

two distinct eigenvalues

$\Downarrow$

A is diagonalizable

$\longrightarrow$ $\underline{\lambda_1 = 6}$, $\lambda_2 = \underline{-1}$

Eigenvectors:

$\lambda = 6$: $\left(\begin{array}{cc|c} 4-6 & -2 & 0 \\ -5 & 1-6 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & -2 & 0 \\ -5 & -5 & 0 \end{array}\right)$

$-2x - 2y = 0 \Rightarrow x = -y$

y free

$E_6 = \text{span}(\underline{v_1})$ $\longrightarrow$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = -1$: $\left(\begin{array}{cc|c} 4+1 & -2 & 0 \\ -5 & 1+1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 5 & -2 & 0 \\ -5 & 2 & 0 \end{array}\right)$

$5x - 2y = 0 \Rightarrow x = \frac{2}{5}y$

y free

$E_{-1} = \text{span}(\underline{v_2})$ $\qquad$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5}y \\ y \end{pmatrix} = \frac{y}{5}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow \underline{v_2} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

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Note: could also have used $\underline{v_2} = \begin{pmatrix} 2/5 \\ 1 \end{pmatrix}$, it gives the same span and the same solutions E_{-1}.

Concl: $\lambda_1 = 6$, $\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$, $\underline{v_2} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\Downarrow$

General solution:

$$\begin{pmatrix} y \\ z \end{pmatrix} = c_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + c_2 \cdot \underline{v_2}\, e^{\lambda_2 t}$$

$$= c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{-t}$$

Can be written:
$$y = -c_1 e^{6t} + 2c_2 e^{-t}$$
$$z = c_1 e^{6t} + 5c_2 e^{-t}$$

b) Eq. states: $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Since $|A| = 4 - 10 = -6 \neq 0$, there is only the trivial solution $\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Downarrow$

$\underline{y_e} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ eq. state.

Stability: $t \to \infty$ means that $e^{6t} \to \infty$ (but $e^{-t} \to 0$)

Therefore $\underline{y_e} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable

For example: $c_2 = 0$, $c_1 = a > 0$ small positive number gives $y(t) \to -\infty$, $z(t) \to \infty$

This corresponds to $\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} -a \\ a \end{pmatrix}$, which is close to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when a is small.