1.1

$$\left.\begin{array}{l} y = t - t^2 \\ y' = 1 - 2t \end{array}\right\}$$

LS: $y' + y = (1-2t) + (t - t^2) = 1 + t - 2t - t^2$
$= 1 - t - t^2$

RS: $1 - t - t^2$ ok, it is a solution

1.2

$$\left.\begin{array}{l} y = e^{rt} \\ y' = e^{rt} \cdot (rt)' = e^{rt} \cdot r \end{array}\right\}$$

LS: $3(y') + 6(y)$
$= 3(re^{rt}) + 6(e^{rt})$
$= (3r + 6)e^{rt}$

RS: 0 equality if and only if $3r + 6 = 0$

$3r + 6 = 0 \iff r = -2$

1.3

a) $y' = 4t^3 + 1 \Rightarrow y = \int 4t^3 + 1 \, dt = t^4 + t + C$

b) $ty' = 2\ln t$
$y' = \frac{2\ln t}{t} = 2\ln t \cdot \frac{1}{t} \Rightarrow y = \int 2\ln t \cdot \frac{1}{t} dt$

$u = \ln t$
$du = \frac{1}{t} dt$

$= \int 2u \cdot du = u^2 + C$

$y = (\ln t)^2 + C$

c) $y' + t^3 = t^2$
$y' = t^2 - t^3 \Rightarrow y = \int t^2 - t^3 \, dt = \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$

d) $e^t y' = t \Rightarrow y = \int te^{-t} dt = -te^{-t} - \int 1 \cdot (-e^{-t}) dt$
$y' = \frac{t}{e^t} = te^{-t}$

$u = -e^{-t}$ $v = t$
$u' = e^{-t}$ $v' = 1$

$= -te^{-t} + (-e^{-t}) + C$
$= -te^{-t} - e^{-t} + C$

1.4 $y'' = 12t + 6$

$$y' = \int (12t+6)\,dt = 12\cdot\frac{1}{2}t^2 + 6\cdot t + C = \underline{6t^2 + 6t + C}$$

$$y = \int (6t^2 + 6t + C)\,dt = 6\cdot\frac{1}{3}t^3 + 6\cdot\frac{1}{2}t^2 + C\cdot t + D$$

$$y = \underline{\underline{2t^3 + 3t^2 + Ct + D}}$$

it depends on C, D – two undetermined coeff.'s (one for each integral)

1.5 $y = Ce^{rt}$, $y' = C\cdot e^{rt}\cdot r$

LS: $y' = rCe^{rt}$

RS: $ry = rCe^{rt}$

ok, it is a solution

1.6 $p' = k\cdot(d-s)$: Assumption is that the rate of change of the price $p(t)$ is proportional with $d-s$

$d-s>0$: price increase proportional to demand surplus $d-s$

$d-s=0$: price stationary

$d-s<0$: price decrease prop. to supply surplus $s-d$

1.7.

General solution of

$$y' = ry\,(1 - y/K) \longrightarrow y = K \cdot \frac{Ce^{rt}}{1+Ce^{rt}}$$

Data: $K = 80$

$y(0) = 56.3$

$y(20) = 58.9$

$y(0) = 56.3$:

$$56.3 = 80 \cdot \frac{C \cdot 1}{1+C \cdot 1} = 80 \frac{C}{1+C} \to \frac{56.3}{80} = \frac{C}{1+C}$$

$$C = (1+C)\frac{56.3}{80} \Rightarrow C = \frac{56.3/80}{1 - 56.3/80} = \frac{56.3}{80 - 56.3}$$

$$C \simeq \underline{2.376}$$

$y(20) = 58.9$:

$$58.9 = 80 \cdot \frac{Ce^{20r}}{1+Ce^{20r}}$$

same steps as above

$$\to Ce^{20r} = \frac{58.9}{80-58.9} \to e^{20r} = \frac{58.9}{80-58.9} \cdot \frac{80-56.3}{56.3}$$

$$\simeq 1{,}175$$

$$20r \simeq \ln(1{,}175)$$

$$r \simeq \frac{\ln(1{,}175)}{20} \simeq \underline{0{,}00807}$$

Solution:

$$\boxed{y = 80 \cdot \frac{2.376 \cdot e^{0{,}00807t}}{1 + 2.376\, e^{0{,}00807t}}}$$

1.8 $y' = 3t^2+6, \quad y(1)=1$

$y = \int 3t^2+6\,dt = t^3+6t+C$

$y(1)=1: \quad 1 = 1^3+6\cdot 1+C$

$1 = 7+C$

$\underline{C=-6}$ $\qquad y(t) = \underline{\underline{t^3+6t-6}}$

1.9 $y' = 3\sqrt{t}, \quad y(0)=1$

$y = \int 3\sqrt{t}\,dt = \int 3\cdot t^{1/2}\,dt = 3\cdot\frac{2}{3}t^{3/2}+C$

$y = 2t^{3/2}+C = 2t\sqrt{t}+C$

$y(0)=1: \quad 1 = 2\cdot 0\cdot\sqrt{0}+C$

$\underline{C=1}$ $\qquad y(t) = \underline{\underline{2t\sqrt{t}+1}}$

1.10. $t^2y' - ty = t+ty$

$t^2y' = ty+t+ty$

$y' = \underline{\underline{\dfrac{ty+t+ty}{t^2}}}$

$F(t,y) = \dfrac{ty+t+ty}{t^2}$

not on form $f(t)$,

So it is not solvable

by single integration

1.11. $ty' = 2\ln t, \quad y(1)=3$

$y' = 2\ln t\cdot\frac{1}{t} \Rightarrow y = \int 2\ln t\,\frac{1}{t}\,dt = \int 2u\,du = u^2+C$

$\boxed{\begin{matrix} u=\ln t \\ du = 1/t\,dt\end{matrix}}$

$y(t) = (\ln t)^2+C$

$\underline{y(1)=3:} \quad 3 = (\ln 1)^2+C = 0+C$

$\underline{C=3} \longrightarrow \quad y(t) = \underline{\underline{(\ln t)^2+3}}$

1.12. $y' = ry = r \cdot y$

$\frac{1}{y} y' = r$

$\int \frac{1}{y} y' \, dt = \int r \, dt$

$\ln|y| = rt + C$

$|y| = e^{rt+C} = e^{C} \cdot e^{rt}$

$y = \pm e^{C} e^{rt} = \underline{\underline{K \cdot e^{rt}}}$ $\quad (K = \pm e^{C})$

1.13.

a) $yy' = t$: $\quad y' = \frac{t}{y} = t \cdot \frac{1}{y} = \underset{\substack{\shortparallel \\ t}}{f(t)} \cdot \underset{\substack{\shortparallel \\ 1/y}}{g(y)}$ $\quad$ separable

$yy' = t$

$\int y \, dy = \int t \, dt \Rightarrow \frac{1}{2} y^2 = \frac{1}{2} t^2 + C$

$y^2 = t^2 + 2C$

$y = \pm \sqrt{t^2 + 2C} = \underline{\underline{\pm \sqrt{t^2 + K}}}$

b) $y' + y = e^t$: $\quad y' = e^t - y \quad \leftarrow$ cannot be factored as $f(t) \cdot g(y) = e^t - y$

not separable

c) $e^y y' = t+1$: $\quad y' = \frac{t+1}{e^y} = \underset{\substack{\uparrow \\ f(t)}}{(t+1)} \cdot \underset{\substack{\uparrow \\ g(y)}}{e^{-y}}$ $\quad$ Separable

$\int e^y \, dy = \int t+1 \, dt \Rightarrow e^y = \frac{1}{2} t^2 + t + C$

$y = \underline{\underline{\ln\left(\frac{1}{2} t^2 + t + C\right)}}$

d) $ty' + y^2 = 1$: $\quad ty' = 1 - y^2 \Rightarrow y' = \frac{1-y^2}{t} = \frac{1}{t}\cdot(1-y^2)$

separable

$$\frac{1}{1-y^2}\, y' = \frac{1}{t}$$

partial fractions

$1 - y^2 = (1-y)(1+y)$

$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y} \qquad \cdot (1-y^2)$

$1 = A(1+y) + B(1-y)$

$= \underbrace{(A+B)}_{=1} + y\cdot\underbrace{(A-B)}_{=0}$

$A = B, \quad A + B = 2A = 1$

$A = 1/2,\ B = 1/2$

$\frac{1}{1-y^2} = \frac{1/2}{1-y} + \frac{1/2}{1+y}$

$$\int \frac{1}{1-y^2}\, dy = \int \frac{1}{t}\, dt$$

$$\int \frac{1/2}{1-y} + \frac{1/2}{1+y}\, dy = \ln|t| + C$$

$$-\frac{1}{2}\ln|1-y| + \frac{1}{2}\ln|1+y| = \ln|t| + C$$

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \ln|t| + C$$

$$\ln\left|\frac{1+y}{1-y}\right| = 2\ln|t| + 2C$$

$$\frac{1+y}{1-y} = \pm e^{2\ln|t| + 2C}$$

$$= \pm e^{2C}\cdot e^{\ln|t|\cdot 2} = \pm e^{2C}\cdot|t|^2 = K\cdot t^2$$

$K = \pm e^{2C}$

$|t|^2 = t^2$

$$1 + y = Kt^2\cdot(1-y)$$

$$y\cdot(1 + Kt^2) = Kt^2 - 1 \Rightarrow y = \frac{Kt^2-1}{Kt^2+1}$$

e) $y' - \ln(t) = 1$: $\quad y' = 1 + \ln(t) = \underbrace{(1+\ln(t))}_{f(t)}\cdot\underbrace{1}_{g(y)}$ separable

$$1\cdot y' = 1 + \ln t$$

$$\int 1\, dy = \int 1 + \ln t\, dt$$

$$y = t + t\ln t - t + C = t\ln t + C$$

1.14. $y' = ry\left(1 - \frac{y}{K}\right)$

$$y' = ry\cdot\left(\frac{K-y}{K}\right) = \underbrace{\frac{r}{K}}_{\substack{f(t)\\(\text{const.})}} \cdot \underbrace{y(K-y)}_{g(y)}$$

$$\frac{1}{y(K-y)}\, y' = \frac{r}{K} \quad \Big| \cdot K$$

$$\frac{K}{y(K-y)}\, y' = r \implies \int \frac{K}{y(K-y)}\, dy = \int r\, dt$$

partial fractions

$$\frac{K}{y(K-y)} = \frac{A}{y} + \frac{B}{K-y} \quad \Big| \cdot y(K-y)$$

$$K = A(K-y) + By$$

$$= \underbrace{AK}_{=K} + \underbrace{(B-A)y}_{=0}$$

$AK = K$ $\quad$ $B - A = 0$

$\underline{A = 1}$ $\quad$ $B = A = \underline{1}$

$$\frac{K}{y(K-y)} = \frac{1}{y} + \frac{1}{K-y}$$

$$\int \frac{1}{y} + \frac{1}{K-y}\, dy = rt + C_1$$

$$\ln|y| - \ln|K-y| = rt + C_1$$

$$\ln\left|\frac{y}{K-y}\right| = rt + C_1$$

$$\frac{y}{K-y} = \pm e^{rt+C_1} = \pm e^{C_1} e^{rt}$$

$$\frac{y}{K-y} = C \cdot e^{rt} \quad (C = \pm e^{C_1})$$

$$y = Ce^{rt}(K-y)$$

$$y\cdot(1 + Ce^{rt}) = K \cdot Ce^{rt}$$

$$\underline{\underline{y = K \cdot \frac{Ce^{rt}}{1 + Ce^{rt}}}}$$

1.15 $\quad y' = ry$

separable: $y' = \underset{f(t)}{r} \cdot \underset{g(y)}{y}$ ok.

linear: $y' - ry = 0$ ok.

$a(t) = -r \quad b(t) = 0$

Integrating factor:

$$\int -r\,dt = -rt + C \longrightarrow u = \underline{e^{-rt}}$$ int. factor

$$y' = ry$$

$$y' - ry = 0 \quad | \cdot e^{-rt}$$

$$y' e^{-rt} - rye^{-rt} = 0$$

$$(y \cdot e^{-rt})' = 0$$

$$y \cdot e^{-rt} = C$$

$$y = \underline{\underline{C \cdot e^{+rt}}} = \underline{\underline{C \cdot e^{rt}}}$$

1.16.

a) $y'+y=e^t \quad |\cdot e^t$ ← linear with int. factor $u=e^{\int 1\,dt}=e^t$

$(ye^t)'=e^{t+t}=e^{2t}$

$ye^t=\int e^{2t}dt=\frac{1}{2}e^{2t}+C$

$y=\left(\frac{1}{2}e^{2t}+C\right)e^{-t}=\underline{\underline{\frac{1}{2}e^t+C\cdot e^{-t}}}$

b) $yy'=t$

$y'=t/y$ not linear

c) $y'=2t+y$

$y'-y=2t$ linear with

$u=e^{\int -1\,dt}=e^{-t}$

$(y\cdot e^{-t})'=2te^{-t}$

$y\cdot e^{-t}=\int 2te^{-t}dt=$

$\boxed{\begin{array}{ll} u'=e^{-t} & v=2t \\ u=-e^{-t} & v'=2 \end{array}}$

$=-2te^{-t}-\int -e^{-t}\cdot 2\,dt$

$=-2te^{-t}+2(-e^{-t})+C$

$y\cdot e^{-t}=-2te^{-t}-2e^{-t}+C$

$y=(-2te^{-t}-2e^{-t}+C)e^t=\underline{\underline{-2t-2+Ce^t}}$

d) $t^2 y' + \ln(t) y = \ln(t)$

$$y' + \frac{\ln t}{t^2} y = \frac{\ln t}{t^2} \quad | \cdot u$$

linear, with

$$\int \frac{\ln t}{t^2} dt = \int \frac{1}{t^2} \cdot \ln t$$

$$= -\frac{1}{t} \cdot \ln t - \int -\frac{1}{t} \cdot \frac{1}{t} dt$$

$$= -\frac{1}{t} \ln t + \left(-\frac{1}{t}\right) + c$$

$$\Downarrow$$

$$u = e^{-\frac{1}{t}\ln t - \frac{1}{t}}$$

$$\left(y \cdot e^{-\frac{1}{t}\ln t - \frac{1}{t}}\right)' = \frac{\ln t}{t^2} e^{-\frac{1}{t}\ln t - \frac{1}{t}}$$

$$y \cdot e^{-\frac{1}{t}\ln t - \frac{1}{t}} = \int \frac{\ln t}{t^2} e^{-\frac{1}{t}\ln t - \frac{1}{t}} dt$$

$$= \int e^v dv$$

Substitution:

$$\boxed{\begin{aligned} v &= -\frac{1}{t}\ln t - \frac{1}{t} \\ dv &= v' \cdot dt \\ &= \frac{\ln t}{t^2} dt \end{aligned}}$$

$$= e^v + C$$

$$y \cdot e^{-\frac{1}{t}\ln t - \frac{1}{t}} = e^{-\frac{1}{t}\ln t - \frac{1}{t}} + C$$

$$y = \left(e^{-\frac{1}{t}\ln t - \frac{1}{t}} + C\right) e^{\frac{1}{t}\ln t + \frac{1}{t}}$$

$$= \underline{\underline{1 + C \cdot e^{\frac{1}{t}\ln t + \frac{1}{t}}}}$$

e) $y' - 2ty = 2t \quad | \cdot e^{-t^2}$ linear

$$\int -2t\, dt = -t^2 + c \Rightarrow u = e^{-t^2}$$

$$\left(y \cdot e^{-t^2}\right)' = 2t e^{-t^2}$$

$$y \cdot e^{-t^2} = \int 2t e^{-t^2} dt = \int e^v \cdot \frac{dv}{-1} \leftarrow \boxed{\begin{aligned} v &= -t^2 \\ dv &= -2t\, dt \end{aligned}}$$

$$= -e^v + c = -e^{-t^2} + C$$

$$y = \left(-e^{-t^2} + C\right) e^{t^2} = \underline{\underline{-1 + C \cdot e^{t^2}}}$$

1.17.

a) $y' + 3y = 4e^t$

$$Y = y_h + y_p = C \cdot e^{-3t} + \cancel{4e^{4t}}\; e^t$$

y_h: $\quad y' + 3y = 0 \quad a = 3 \rightarrow y_h = C \cdot e^{-3t}$

y_p: $\quad y = e^t$: $\quad D(y) = e^t + 3e^t = 4e^t \rightarrow y_p = \cancel{4e^t}\; e^t$

ok.

b) $y' - y = t$

$$y = y_h + y_p = Ce^t - t - 1$$

y_h: $\quad y' - y = 0 \quad a = -1 \rightarrow y_h = Ce^t$

y_p: $\quad y = t$: $\quad D(y) = 1 - t$

$y = 1$: $\quad D(y) = 0 - 1$

$y = t + 1$: $\quad D(y) = (1 - t) + (-1) = -t$

$D(-t-1) = t - 1 + 1 = t \rightarrow y_p = -t - 1$

c) $y' = 2t + y$

$y' - y = 2t$

$$y = y_h + y_p = Ce^t + 2t - 2$$

$y_h = Ce^t$ (see above)

y_p: $\quad y = at + b$: $\quad D(at+b) = a - (at + b) = -at + (a - b) = 2t$

$-a = +2 \rightarrow a = -2$

$a - b = 0 \quad b = a = -2$

$y_p = -2t - 2$

d) $6y' - 18y = 12t$ $\qquad y = y_h + y_p = \underline{\underline{Ce^{3t} - \frac{2}{3}t - \frac{2}{9}}}$

y_h: $6y' - 18y = 0$

$y' - 3y = 0 \rightarrow \underline{y_h = C \cdot e^{3t}}$

y_p: $y = At + B$

$D(y) = 6A - 18(At + B) = 12t$

$\underbrace{-18A}_{12}\, t + \underbrace{(6A - 18B)}_{0} = 12t$

$$\left.\begin{array}{l} -18A = 12 \rightarrow A = -12/18 = \underline{-2/3} \\ 6A - 18B = 0 \rightarrow B = \frac{6A}{18} = A/3 = \underline{-2/9} \end{array}\right\} \underline{y_p = -\frac{2}{3}t - \frac{2}{9}}$$

1.18. $t^2 y' + \ln(t) y = \ln(t)$ $\qquad y = y_h + y_p = \underline{\underline{C \cdot e^{\frac{1}{t}\ln t + \frac{1}{t}} + 1}}$

$\underline{y_h}$: $t^2 y' + \ln t \, y = 0$

$y' + \frac{\ln t}{t^2} y = 0 \leftarrow$

$\left(y \cdot e^{-\frac{1}{t}\ln t - \frac{1}{t}}\right)' = 0$

$y \cdot e^{-\frac{1}{t}\ln t - \frac{1}{t}} = C$

$y_h = \underline{C e^{\frac{1}{t}\ln t + \frac{1}{t}}}$

Int. factor since $a(t) = \frac{\ln t}{t^2}$ non-constant

$\int \frac{\ln t}{t^2} dt = \int \frac{1}{t^2} \ln(t) \, dt$

$= -\frac{1}{t} \cdot \ln t - \int -\frac{1}{t} \cdot \frac{1}{t} \, dt$

$= -\frac{1}{t} \ln t - \frac{1}{t} + C$

$u = e^{-\frac{1}{t}\ln t - \frac{1}{t}}$

y_p: $y = 1 \rightarrow D(1) = 0 + \ln t = \ln t$

$y_p = \underline{1}$

1.19. $ty' + 2y = t$, $y(1) = 1$

$$\boxed{y' + \frac{2}{t}y = 1}$$ linear diff. eqn. in std. form

$$y = y_h + y_p = c \cdot \frac{1}{t^2} + \frac{1}{3}t$$

y_h: $y' + \frac{2}{t}y = 0$ use integrating factor

$u = e^{\int 2/t\, dt} = e^{2\ln|t| + c}$

$= |t|^2 = t^2$ ← use $c = 0$

$(t^2 \cdot y)' = t^2 \cdot 0 = 0$

$t^2 y = \int 0\, dt = c$

$y = c/t^2 = c \cdot \frac{1}{t^2}$

y_p: try $y = At$ (A const) $\Rightarrow y' = A$:

$y' + \frac{2}{t}y = 1$

$A + \frac{2}{t} \cdot (At) = 1$

$A + 2A = 1$

$3A = 1$

$y_p = \frac{1}{3}t \Leftarrow A = 1/3$

Alternative solution:

Find the general solution by using integrating factor on

$y' + \frac{2}{t}y = 1$ Int. factor: $u = t^2$

$(y \cdot u)' = 1 \cdot t^2$

$\frac{y \cdot u}{u} = \int t^2 dt = \frac{\frac{1}{3}t^3 + c}{u} = \frac{\frac{1}{3}t^3 + c}{t^2}$

$y = \frac{1}{3}t + \frac{c}{t^2}$

Particular solution:

General Solution: $y' + \frac{2}{t} y = 1 \Rightarrow y = C \cdot \frac{1}{t^2} + \frac{1}{3} t$

$y(1) = 1$: $1 = C \cdot \frac{1}{1} + \frac{1}{3} \cdot 1$

$1 = C + 1/3$

$C = 1 - 1/3 = \underline{2/3}$

$$y = \frac{2}{3} \cdot \frac{1}{t^2} + \frac{1}{3} t$$

$$= \underline{\underline{\frac{1}{3}\left(\frac{2}{t^2} + t\right)}}$$

1.20. $(1+2ty^2)+2t^2y\cdot y'=0$

$h'_t=1+2ty^2 \Rightarrow h=t+t^2y^2+a(y)$

$\Rightarrow h'_y=0+t^2 2y+a'(y) = 2t^2y$

$a'(y)=0$

$a(y)=0$

$\Downarrow$

$h=t+t^2y^2$ gives $1+2ty^2+2t^2yy'=0$

$h'_t+h'_y\cdot y'=0$

$h=C$

$t+t^2y^2=C$

$t^2y^2=C-t$

$y^2=\frac{C-t}{t^2}$

$y=\pm\sqrt{\frac{C-t}{t^2}}$

$y(1)=-1$: $t+t^2y^2=C$

$1+1^2\cdot(-1)^2=C$

$C=2 \Rightarrow y(t)=-\sqrt{\frac{2-t}{t^2}}$

1.21.

a) $2t-y+(2y-t)y'=0$

$h'_t=2t-y \Rightarrow h=t^2-yt+a(y) \Rightarrow h'_y=0-t+a'(y)=2y-t$

$a'=2y$

$a=y^2$

$h=t^2-yt+y^2=C$

$y^2-ty+(t^2-C)=0$

$y=\frac{t\pm\sqrt{t^2-4(t^2-C)}}{2}$

or $y=\frac{t}{2}\pm\frac{1}{2}\sqrt{4C-3t^2}$

b) $ye^t + e^t y' = 0$

$$h'_t = ye^t \Rightarrow h = ye^t + \varphi(y) \Rightarrow h'_y = e^t + \varphi'(y) = e^t$$

$$\varphi'(y) = 0$$

$$\varphi(y) = 0$$

$$h = ye^t = C$$

$$\underline{\underline{y = C \cdot e^{-t}}}$$

c) $ty^2 + y + (t^2 y + t)y' = 0$

$$h'_t = ty^2 + y \Rightarrow h = \frac{1}{2}t^2 y^2 + yt + \varphi(y)$$

$$\Rightarrow h'_y = t^2 \cdot y + t + \varphi'(y) = t^2 y + t$$

$$\varphi'(y) = 0$$

$$\varphi(y) = 0$$

$$h = \frac{1}{2}t^2 y^2 + ty = C$$

$$t^2 y^2 + 2ty - 2C = 0$$

$$y = \frac{-2t \pm \sqrt{4t^2 - 4t^2(-2C)}}{2t^2}$$

$$= -\frac{1}{t} \pm \frac{\sqrt{4t^2(1+2C)}}{2t^2}$$

$$\underline{\underline{= -\frac{1}{t} \pm \frac{\sqrt{1+2C}}{t}}}$$

1.22. Separable: $y' = f(t)\cdot g(y)$

$$\frac{1}{g(y)}\cdot y' = f(t)$$

$$-f(t) + \frac{1}{g(y)}\cdot y' = 0$$

Exact: $\left.\begin{array}{l} h'_t = -f(t) \\ h'_y = 1/g(y) \end{array}\right\}$

$$h = \int \frac{1}{g(y)}\,dy - \int f(t)\,dt$$ Satisfies,

therefore the equation is exact.

1.23. Linear: $y' + a(t)\cdot y = b(t)$

Int. factor: $u = e^{\int a(t)dt}$

$$uy' + a(t)\cdot uy = u(t)b(t)$$

$$(y\cdot u)' = u(t)\cdot b(t)$$

$$-u(t)b(t) + u(t)y' + \underbrace{a(t)u(t)}_{u'}y = 0$$

$\left.\begin{array}{l} h'_t = u'(t)y - u(t)b(t) \\ h'_y = u(t) \end{array}\right\}$ Satisfies

$\Downarrow$

$$h = u(t)\cdot y - \int u(t)b(t)\,dt$$

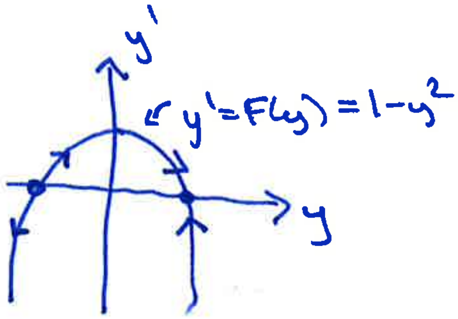
therefore the ~~equation~~ equation is exact.

1.24. $y' = 1 - y^2$

$F(y) = 1 - y^2 \Rightarrow F(y) = 0$
$1 - y^2 = 0$
$y = \pm 1$

Hence the eq. states are

$y_e = -1$, $y_e = 1$



With y_0 close to $y = -1$:

$y_0 > -1 \Rightarrow y' = F(y) > 0 \Rightarrow y(t)$ increases when t incr.

$y_0 < -1$ $\quad y' = F(y) < 0$ $\quad y(t)$ decreases when t decr.

$F(y) = 1 - y^2$

$F'(y) = -2y$

$F'(-1) = 2 > 0 \Rightarrow y_e = -1$ is unstable

$F'(1) = -2 < 0$

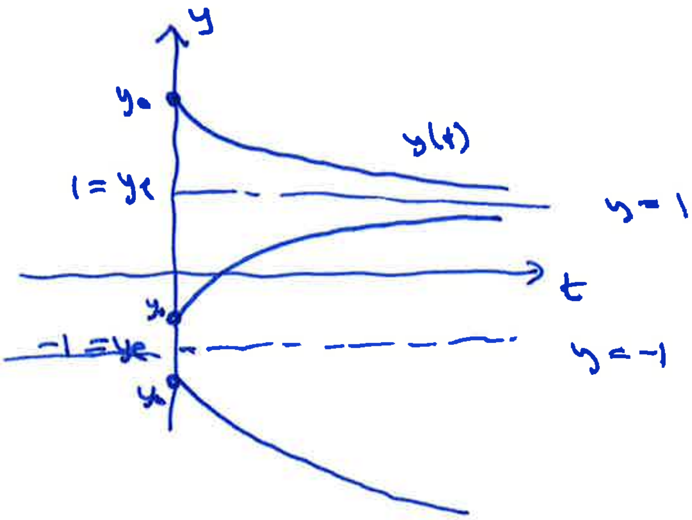
With y_0 close to $y = 1$:

$y_0 > 1 \Rightarrow F(y) = y' < 0$ $\quad y(t)$ decreases when t incr.

$y_0 < 1$ $\quad F(y) = y' > 0$ $\quad y(t)$ increases when t incr.

$\Rightarrow y_e = 1$ is stable

Since $y_0 < -1$ will force $y(t)$ to move away from $y_e = -1$ and therefore away from $y_e = 1$ $(y \to -\infty)$, the stable eq. state $y = 1$ is not globally asymptotically stable.



1.25.

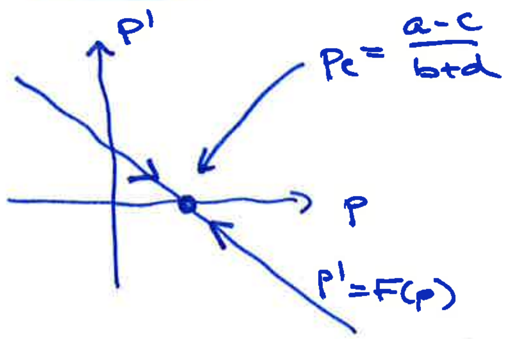
$$p' = k(d-s)$$
$$= k\big((a-bp)-(c+dp)\big)$$
$$= k(a-c) - k(b+d)p$$

$$p' = F(p) \quad \text{with} \quad F(p) = \underbrace{k(a-c)}_{\text{const.}} - \underbrace{k(b+d)}_{\text{const.}}p$$

Eq. state: $F(p)=0 \Leftrightarrow k(a-c) - k(b+d)p = 0$

$$k(a-c) = k(b+d)p$$
$$p_e = \frac{\cancel{k}(a-c)}{\cancel{k}(b+d)} = \frac{a-c}{b+d}$$

Stability:



$$F(p) = k(a-c) - k(b+d)p$$
$$F'(p) = -k(b+d) < 0$$
$$\Downarrow$$
Straight line with negative slope

p_e is stable since

$p_0 > p_e \Rightarrow p(t)$ decreases $(F(p) = p' < 0)$

$p_0 < p_e \Rightarrow p(t)$ increases $(F(p) = p' > 0)$

$$p(t) = \frac{a-c}{b+d} + C \cdot e^{-k(b+d)t}$$
with $C = p_0 - \frac{a-c}{b+d}$

$$\Downarrow$$

$\lim_{t\to\infty} p(t) = \frac{a-c}{b+d} = p_e$ for all $p_0 \Rightarrow p_e$ is globally asymptotically stable

1.26.

$$y'' = 0 \Rightarrow y' = C \Rightarrow y = \int C\,dt = \underline{Ct + D}$$

1.27.

$$y'' = e^{t} - e^{-t} \Rightarrow y' = \int e^{t} - e^{-t}\,dt = e^{t} + e^{-t} + C$$

$$\Rightarrow y = \int e^{t} + e^{-t} + C\,dt = \underline{e^{t} - e^{-t} + Ct + D}$$

1.28.

$$y'' = 1 - y'$$

$\underline{z = y'}$: $z' = 1 - z$

$z' + z = 1$ linear first order

$\Downarrow$

$$z = z_h + z_p = \underline{C \cdot e^{-t} + 1}$$

$\underline{z_h}$: $z' + z = 0 \Rightarrow z_h = \frac{b}{a} + Ce^{-at} = \underline{Ce^{-t}}$

$\underline{z_p}$: $z = A$ constant $\Downarrow$ $z' = 0$: $z' + z = 1$

$0 + A = 1$

$\underline{A = 1} \Rightarrow \underline{z_p = 1}$

$$y' = z = Ce^{-t} + 1$$

$$y = \int Ce^{-t} + 1\,dt = -Ce^{-t} + t + D$$

$$y = \underline{\underline{t - Ce^{-t} + D}}$$

1.29. $y'' + y' - 2y = 4t$, $y(0) = 1$, $y'(0) = 0$

$y = y_h + y_p = c_1 e^{-2t} + c_2 e^{t} - 2t - 1$

y_h: $y'' + y' - 2y = 0$

$r^2 + r - 2 = 0$

$r = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$

$r = -2$, $r = 1$ $\rightarrow$ $y_h = \underline{c_1 e^{-2t} + c_2 e^{t}}$

y_p: $y'' + y' - 2y = 4t$

$h(t) = 4t$ Guess: $y = At + B$

$h' = 4$ $y' = A$

$h'' = 0$ $y'' = 0$

$0 + A - 2(At + B) = 4t$

$(-2A)t + (A - 2B) = 4t$

$-2A = 4$, $A - 2B = 0$

$A = -2$ $B = -1$ $\longrightarrow$ $y_p = \underline{-2t - 1}$

$y = c_1 e^{-2t} + c_2 e^{t} - 2t - 1$

$y' = -2c_1 e^{-2t} + c_2 e^{t} - 2$

$y(0) = 1$: $c_1 + c_2 - 1 = 1 \Rightarrow c_1 + c_2 = 2 \Rightarrow \underline{c_2 = 2 - c_1}$

$y'(0) = 0$: $-2c_1 + c_2 - 2 = 0 \Rightarrow -2c_1 + (\cancel{2} - c_1) - \cancel{2} = 0$

$-3c_1 = 0 \Rightarrow \underline{c_1 = 0}$

$\underline{c_2 = 2}$

$y = \underline{\underline{2e^{t} - 2t - 1}}$

1.30.

a) $y'' - 4y = t+1 \Rightarrow y = y_h + y_p = C_1 e^{-2t} + C_2 e^{2t} - \frac{1}{4}t - \frac{1}{4}$

y_h: $r^2 - 4 = 0$

$r = \pm 2 \rightarrow y_h = C_1 e^{-2t} + C_2 e^{2t}$

y_p: $y'' - 4y = t+1$

$$\left.\begin{array}{l} h = t+1 \\ h' = 1 \\ h'' = 0 \end{array}\right\} \quad \text{Guess: } \begin{array}{l} y = At+B \\ y' = A \\ y'' = 0 \end{array}$$

$-4(At+B) = t+1$

$-4At - 4B = t+1$

$-4A = 1 \qquad -4B = 1 \qquad y_p = -\frac{1}{4}t - \frac{1}{4}$

$A = -1/4 \qquad B = -1/4$

b) $y'' + 3y' = e^{-t} \Rightarrow y = y_h + y_p = C_1 e^{-3t} + C_2 - \frac{1}{2}e^{-t}$

y_h: $r^2 + 3r = 0$

$r = 0, -3 \rightarrow y_h = C_1 e^{-3t} + C_2 e^{0t} = C_1 e^{-3t} + C_2$

y_p: $y'' + 3y' = e^{-t}$

$$\left.\begin{array}{l} h = e^{-t} \\ h' = -e^{-t} \\ h'' = +e^{-t} \end{array}\right\} \quad \text{Guess: } \begin{array}{l} y = A \cdot e^{-t} \\ y' = -A e^{-t} \\ y'' = A e^{-t} \end{array}$$

$(Ae^{-t}) + 3(-Ae^{-t}) = e^{-t}$

$(A - 3A)e^{-t} = e^{-t}$

$-2A = 1$

$A = -1/2 \rightarrow y_p = -\frac{1}{2}e^{-t}$

c) $y'' + 5y' - 6y = t^2 \implies y = y_h + y_p = C_1 e^{-6t} + C_2 e^{t} - \frac{1}{6}t^2 - \frac{5}{18}t - \frac{31}{108}$

y_h: $y'' + 5y' - 6y = 0$

$r^2 + 5r - 6 = 0$

$(r+6)(r-1) = 0$

$r = -6, \; r = 1 \implies y_h = C_1 e^{-6t} + C_2 e^{t}$

y_p: $y'' + 5y' - 6y = t^2 \longrightarrow \left. \begin{array}{l} h = t^2 \\ h' = 2t \\ h'' = 2 \end{array} \right\}$ Guess: $\begin{array}{l} y = At^2 + Bt + C \\ y' = 2At + B \\ y'' = 2A \end{array}$

$2A + 5(2At + B)$
$- 6(At^2 + Bt + C) = t^2$

$(-6A)t^2 + (10A - 6B)t + (2A + 5B - 6C) = t^2$

$-6A = 1$
$A = -1/6$

$10A - 6B = 0$
$6B = 10A = -10/6$
$B = -10/36 = -5/18$

$2A + 5B - 6C = 0$
$6C = 2A + 5B = 2 \cdot (-1/6) + 5(-5/18)$
$= \frac{-6-25}{18} = \frac{-31}{18}$
$C = -\frac{31}{18 \cdot 6} = -\frac{31}{108}$

$y_p = -\frac{1}{6}t^2 - \frac{5}{18}t - \frac{31}{108}$

1.31.

$$y'' + 3y' - 4y = 2e^t$$

$$y = y_h + y_p = \underline{\underline{C_1 e^{-4t} + C_2 e^t + \frac{5}{2} t e^t}}$$

y_h:

$$y'' + 3y' - 4y = 0$$

$$r^2 + 3r - 4 = 0$$

$$r = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2}$$

$$r = \underline{-4}, \quad r = \underline{1} \quad \Rightarrow y_h = \underline{C_1 \cdot e^{-4t} + C_2 e^t}$$

$\underline{y_p}$:

$$y'' + 3y' - 4y = 2e^t$$

$$\left.\begin{aligned} h(t) &= 2e^t \\ h' &= 2e^t \\ h'' &= 2e^t \end{aligned}\right\}$$

Choose

$$y = A \cdot e^t$$

$$\Downarrow$$

$$y' = Ae^t$$

$$y'' = Ae^t$$

$$Ae^t + 3Ae^t - 4Ae^t = 2e^t$$

$$0 \cdot e^t = 2 \cdot e^t \leftarrow \text{no solution}$$

$$\underline{\underline{t}} \cdot Ae^t$$

Try:

$$y = Ate^t$$

$$y' = A \cdot e^t + At \cdot e^t = (At + A)e^t$$

$$y'' = (At+A)e^t + Ae^t = (At + 2A)e^t$$

$$(At+2A)e^t + 3(At+A)e^t - 4(Ate^t) = 2e^t$$

$$(0) \cdot te^t + (5A)e^t = 2e^t$$

$$5A = 2$$

$$A = 2/5 \rightarrow y_p = Ate^t = \underline{\underline{\frac{2}{5} te^t}}$$

1.32.

Assume:

$a^2-4b=0$ such that $r=-a/2$ is a double root of $r^2+ar+b=0$.

$$y''+ay'+by=0 \quad \text{diff. eqn.}$$
$$\Updownarrow$$
$$r^2+ar+b=0 \quad \text{char. eqn.}$$
$$\Updownarrow$$
$$r=\frac{-a\pm\sqrt{a^2-4b}}{2}$$

double root
$\Updownarrow$
$a^2-4b=0$

$$y=t\cdot e^{rt} \qquad r=-\frac{a}{2}$$
$$y'=1\cdot e^{rt}+t\cdot e^{rt}\cdot r = (1+rt)e^{rt}$$
$$y''=r\cdot e^{rt}+(1+rt)\cdot e^{rt}\cdot r = (r^2t+2r)e^{rt} = (r^2t+2r)e^{rt}$$

$$y''+ay'+by$$
$$\longrightarrow = (r^2t+2r)e^{rt}+a(1+rt)e^{rt} + bte^{rt}$$
$$=(r^2t+art+bt + 2r+a)\cdot e^{rt}$$
$$=(r^2+ar+b)\cdot te^{rt} + (2r+a)\cdot e^{rt} = \underline{0}$$

since:

* $r^2+ar+b=0$ since $r=-\frac{a}{2}$ is a characteristic root
* $2r+a=0$ since $r=\frac{-a}{2}$

This means that $y=t\cdot e^{rt}$ is a solution of $y''+ay'+by=0$ when $r=-a/2$ is a double root of the characteristic equation.

1.3~~3~~. $y' - 4y = te^t$

(a)   Superposition principle (for linear first order diff.-eqn.) :

$$y = y_h + y_p = \underline{\underline{C_1 \cdot e^{4t} + \left(-\tfrac{1}{3}t - \tfrac{1}{9}\right)e^t}}$$

$\underline{y_h}$: $y' - 4y = 0$ Characteristic equation.

$r - \cancel{4r} = 0$

$\underline{r = 4}$ $\longrightarrow$ $y_h = \underline{C_1 \cdot e^{4t}}$

$\underline{y_p}$: $y' - 4y = te^t$

$h = te^t$ $\Rightarrow$ $\underline{\text{Guess:}}$

$h' = 1 \cdot e^t + t \cdot e^t$
$= (t+1)e^t$
$h'' = 1 \cdot e^t + (t+1)e^t$
$= (t+2)e^t$

$y = \underline{(At+B)}e^t$
$y' = A \cdot e^t + (At+B)e^t$
$= \underline{(At+A+B)e^t}$

$y'' = Ae^t + (At+A+B)e^t$
$= \underline{(At+2A+B)e^t}$

we see that we didn't really have to compute h'' and y'' in the first order case

$(At+A+B)e^t$
$-4(At+B)e^t = te^t$

$(-3A)te^t + (A-3B)e^t = te^t$

$-3A = 1$ $A - 3B = 0$
$\underline{A = -1/3}$ $3B = A = -1/3$
$\underline{B = -1/9}$

$\Downarrow$

$$y_p = \left(-\tfrac{1}{3}t - \tfrac{1}{9}\right)e^t$$

(b) <u>Integrating factor:</u>

$$y' - 4y = te^t$$

$a(t) = -4$

$\int a(t)\,dt = -4t + C$

$\underline{u = e^{-4t}}$ is int. factor

$$y' e^{-4t} - 4y e^{-4t} = te^t \cdot e^{-4t}$$

$$(y \cdot e^{-4t})' = t \cdot e^{-3t}$$

$$y \cdot e^{-4t} = \int t \cdot e^{-3t}\,dt = t \cdot \left(-\tfrac{1}{3} e^{-3t}\right) - \int 1 \cdot \left(-\tfrac{1}{3} e^{-3t}\right) dt$$

$$y \cdot e^{-4t} = -\frac{1}{3} te^{-3t} + \int \frac{1}{3} e^{-3t}\,dt$$

$$= -\frac{1}{3} te^{-3t} - \frac{1}{9} e^{-3t} + C$$

$$y = e^{4t}\left(-\frac{1}{3} te^{-3t} - \frac{1}{9} e^{-3t} + C\right)$$

$$= -\frac{1}{3} te^t - \frac{1}{9} e^t + C \cdot e^{4t}$$

$$\underline{\underline{y = \underbrace{-\frac{1}{3} te^t - \frac{1}{9} e^t}_{y_p} + \underbrace{C \cdot e^{4t}}_{y_h}}}$$

<u>Comparison:</u>

i) Same result: y_p y_h

ii) Integrating factor: more integrals
Superposition principle: more algebra

1.34. $D = \frac{d^2}{dt^2} + a(t)\cdot \frac{d}{dt} + b(t)$

$$= \{ y \longmapsto y'' + a(t)y' + b(t)y = D(y) \}$$

D is linear: ok since it satisfies the requirements

i) $D(y_1 + y_2) = (y_1+y_2)'' + a(t)(y_1+y_2)' + b(t)(y_1+y_2)$

$$= (y_1'' + y_2'') + a(t)\cdot(y_1' + y_2') + b(t)\cdot(y_1+y_2)$$

$$= y_1'' + a(t)y_1' + b(t)y_1 + y_2'' + a(t)y_2' + b(t)y_2$$

$$= D(y_1) + D(y_2)$$

ii) $D(c\cdot y) = (cy)'' + a(t)\cdot(cy)' + b(t)(cy)$

$$= c\cdot y'' + a(t)\cdot c\cdot y' + b(t)\cdot c\cdot y$$

$$= c\cdot(y'' + a(t)\cdot y' + b(t)y) = c\cdot D(y)$$

2.1. $\left.\begin{array}{l} y' = y+z \\ z' = 2z \end{array}\right\}$ (2) $z' = 2z$ is linear (and not coupled)

$\Downarrow$

$z' - 2z = 0 \,|\cdot u \quad \leftarrow \quad u = e^{-2t}$ int. factor

we must de-couple the system

(= find diff eqn. in one variable at the time)

$(u \cdot z)' = 0$

$(e^{-2t} \cdot z)' = 0$

$e^{-2t} \cdot z = C$

$z = C \cdot e^{2t}$

(1) $y' = y + z = y + Ce^{2t}$

$y' - y = Ce^{2t}$ linear (and not coupled)

$(y \cdot u)' = u \cdot Ce^{2t} \leftarrow u = e^{-t}$

$(y \cdot e^{-t})' = e^{-t} \cdot C \cdot e^{2t} = C \cdot e^{t}$

$ye^{-t} = \int Ce^{t}\, dt = Ce^{t} + D$

$y = e^{t}(Ce^{t} + D) = Ce^{2t} + De^{t}$

General solution:

$y = Ce^{2t} + De^{t}$

$z = Ce^{2t}$

$y(0) = 1$: $1 = C \cdot e^{2 \cdot 0} + D \cdot e^{0} = C + D$ $\quad D = 1 - C = -1$

$z(0) = 2$: $2 = C \cdot e^{2 \cdot 0} = C \quad \Rightarrow C = 2$

Particular solution:

$y(t) = 2e^{2t} - e^{t}$

$z(t) = 2e^{2t}$

2.2. $y'' - 7y' + 12y = 4$

use diff. eqn.

Put: $\left.\begin{array}{l} u = y \\ v = y' \end{array}\right\} \Rightarrow \begin{array}{l} u' = y' = v \\ v' = y'' = 7y' - 12y + 4 = 7v - 12u + 4 \end{array}$

System: $\boxed{\begin{array}{l} u' = v \\ v' = 4 - 12u + 7v \end{array}}$

2.3. $y' = Ay$, $A = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 6-\lambda & -3 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 0 = 0$$

$$\lambda(\lambda-7)=0$$

$$\lambda_1 = 0, \ \lambda_2 = 7$$

E_0: $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{matrix} 6y - 3z = 0 \\ -2y + z = 0 \end{matrix} \Rightarrow \begin{matrix} y = \text{free} \\ z = 2y \end{matrix}$$

$E_0 = \text{span}(\underline{v_1})$ $\quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

E_7: $\begin{pmatrix} -1 & -3 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{matrix} -y - 3z = 0 \\ -2y - 6z = 0 \end{matrix}$

$$\Downarrow$$

$E_7 = \text{span}(\underline{v_2})$, $\underline{v_2} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\quad \begin{matrix} y = -3z \\ z = \text{free} \end{matrix}$

$D = \begin{pmatrix} 0 & 0 \\ 0 & 7 \end{pmatrix}$ $\quad P = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$

A is symmetric.

Solution:

$$\underline{y} = c_1 \cdot \underline{u} \cdot e^{0t} + c_2 \cdot \underline{u_2} \cdot e^{7t}$$

$$= c_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot e^0 + c_2 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{7t} = \begin{pmatrix} c_1 - 3c_2 e^{7t} \\ 2c_1 + c_2 e^{7t} \end{pmatrix}$$

2.4. $\underline{y}' = A\underline{y}$, $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = (1-\lambda)\big((4-\lambda)(-\lambda)-4\big) - 2\big(2(-\lambda)-4\big) + 1\cdot\big(4-2(4-\lambda)\big)$$

$$= (1-\lambda)(\lambda^2-4\lambda-4) + 4\lambda+8+2\lambda-4$$

$$= -\lambda^3+5\lambda^2-4 + 6\lambda+4 = -\lambda^3+5\lambda^2+6\lambda$$

$$= -\lambda(\lambda^2-5\lambda-6) = 0$$

$\underline{\lambda=0}$ or $\underline{\lambda=-1}$ or $\underline{\lambda=6}$

$\underline{\lambda_1=0}$: $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x+2y+2z=0$, $-2z=0$ $\quad x=-2y$, $z=0$, y free

$E_0 = \text{span}(\underline{v_1})$, $\underline{v_1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$\underline{\lambda_2=-1}$: $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x=-y-z=-z$, $y=0$, $z=\text{free}$

$E_{-1} = \text{span}(\underline{v_2})$, $\underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda_3=6}$: $\begin{pmatrix} -5 & 2 & 2 \\ 2 & -2 & 2 \\ 1 & 2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -6 \\ 0 & -6 & 14 \\ 0 & 12 & -28 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -6 \\ 0 & 3 & -7 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x = -14/3\,z + 6z = 4/3\,z$, $y = 7z/3$, $z=\text{free}$

$E_6 = \text{span}(\underline{v_3})$; $\underline{v_3} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$

$\Downarrow$

$$\underline{y} = c_1\cdot \underline{v_1}\cdot e^{0t} + c_2\,\underline{v_2}\cdot e^{-t} + c_3\,\underline{v_3}\,e^{6t}$$

$$= c_1\cdot\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_2\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}e^{6t} = \underline{\underline{\begin{pmatrix} -2c_1 - c_2e^{-t} + 4c_3e^{6t} \\ c_1 + 7c_3e^{6t} \\ c_2e^{-t} + 3c_3e^{6t} \end{pmatrix}}}$$

2.5. We have general solution

$$\underline{y} = C_1 \underline{v_1} e^{\lambda_1 t} + \dots + C_n v_n e^{\lambda_n t}$$

If $\lambda_1, \dots, \lambda_n < 0$, then $e^{\lambda_1 t} \to 0, e^{\lambda_2 t} \to 0, \dots, e^{\lambda_n t} \to 0$ as $t \to \infty$. Therefore, $\underline{y} \to \underline{0}$ no matter what the initial conditions are. This means that $\underline{y} = \underline{0}$ is globally asymptotically stable.

2.6. $y'' + 7y' + 12y = 3 \Rightarrow y'' = 3 - 12y - 7y'$

$$\left.\begin{array}{l} u = y \\ v = y' \end{array}\right\} \Rightarrow \quad \begin{array}{l} u' = v \\ v' = 3 - 12u - 7v \end{array}$$

Steady state:

$$v = 0$$
$$3 - 12u - 7v = 0$$
$$\Updownarrow$$
$$v = 0, \quad 3 - 12u = 0$$
$$u = 1/4$$

$y_e = \underline{\underline{1/4}}$ eq. state $\longleftarrow$ $(u, v) = (1/4, 0)$

General solution: $y = y_h + y_p = \underbrace{C_1 e^{-3t} + C_2 e^{-4t}}_{y_h \text{ since}} + \underbrace{1/4}_{y_p}$

$r^2 + 7r + 12 = 0$ gives $r = -3, -4$

$y(t) \to 1/4$ for all C_1, C_2 since $-3, -4 < 0$

$$\Updownarrow$$

$y(t) \to y_e$ for all initial states

$y_e = 1/4$ is globally asymptotically stable

A.1.

a) $\int 3t^2 - 12t \, dt = t^3 - 12 \cdot \frac{1}{2} t^2 + C = \underline{t^3 - 6t^2 + C}$

b) $\int 2e^t - t \, dt = \underline{2e^t - \frac{1}{2} t^2 + C}$

c) $\int t\sqrt{t} \, dt = \int t^{3/2} \, dt = \frac{2}{5} \cdot t^{5/2} + C$

$= \underline{\frac{2}{5} t^2 \sqrt{t} + C}$

d) $\int \frac{1}{t^3} \, dt = \int t^{-3} \, dt = \frac{1}{-2} t^{-2} + C = \underline{-\frac{1}{2} \frac{1}{t^2} + C}$

e) $\int (t-1)^2 \, dt = \int t^2 - 2t + 1 \, dt = \underline{\frac{1}{3} t^3 - t^2 + t + C}$

A.2.

$$\int \frac{t^3 - t^2 + 1}{t} \, dt = \int t^2 - t + 1/t \, dt = \underline{\frac{1}{3} t^3 - \frac{1}{2} t^2 + \ln|t| + C}$$

A.3

$$\boxed{\begin{matrix} u=\frac{1}{2}t^2 & v=\ln t \\ u'=t & v'=1/t \end{matrix}}$$

a) $\int t \ln t \, dt \overset{!}{=} \frac{1}{2}t^2 \cdot \ln t - \int \frac{1}{2}t^2 \cdot \frac{1}{t} \, dt = \frac{1}{2}t^2 \ln t - \frac{1}{2}\int t \, dt$

$$= \frac{1}{2}t^2 \ln t - \frac{1}{2}\cdot\frac{1}{2}t^2 + C = \underline{\frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 + C}$$

$$\boxed{\begin{matrix} u=e^t & v=t \\ u'=e^t & v'=1 \end{matrix}}$$

b) $\int t e^t \, dt \overset{!}{=} te^t - \int e^t \cdot 1 \, dt = te^t - \int e^t \, dt$

$$\underline{= te^t - e^t + C}$$

$$\boxed{\begin{matrix} u=e^t & v=t^2 \\ u'=e^t & v'=2t \end{matrix}}$$

c) $\int t^2 e^t \, dt \overset{!}{=} t^2 e^t - \int 2te^t \, dt = t^2e^t - 2\int te^t \, dt$

from b)

$$= t^2e^t - 2(te^t - e^t) + C = \underline{t^2e^t - 2te^t + 2e^t + C}$$

d) $\int \frac{\ln t}{t^2} \, dt = \int t^{-2} \cdot \ln t \, dt = -\frac{1}{t} \cdot \ln t - \int -\frac{1}{t} \cdot \frac{1}{t} \, dt$

$$\boxed{\begin{matrix} u=-t^{-1} & v=\ln t \\ u'=t^{-2} & v'=1/t \end{matrix}}$$

$$= -\frac{1}{t}\ln t + \int t^{-2} \, dt = \underline{-\frac{1}{t}\ln t - \frac{1}{t} + C}$$

e) $\int \sqrt{t} \ln t \, dt = \int t^{1/2} \cdot \ln t \, dt =$

$$\boxed{\begin{matrix} u=\frac{2}{3}t^{3/2} & v=\ln t \\ u'=t^{1/2} & v'=1/t \end{matrix}}$$

$$= \frac{2}{3}t^{3/2} \cdot \ln t - \int \frac{2}{3}t^{3/2} \cdot \frac{1}{t} \, dt = \frac{2}{3}t\sqrt{t}\ln(t) - \frac{2}{3}\int t^{1/2} \, dt$$

$$= \frac{2}{3}t\sqrt{t}\ln t - \frac{2}{3}\cdot\frac{2}{3}t^{3/2} + C = \underline{\frac{2}{3}t\sqrt{t}\ln t - \frac{4}{9}t\sqrt{t} + C}$$

A4.

$$\int \frac{\ln t}{t}\,dt = \int t^{-1}\cdot \ln t\,dt = \ln t\cdot \ln t - \int \ln t\cdot \frac{1}{t}\,dt$$

$$\boxed{\begin{array}{ll} u = \ln t & v = \ln t \\ u' = t^{-1} & v' = 1/t \end{array}}$$

$$= (\ln t)^2 - \int \frac{\ln t}{t}\,dt$$

Same integral as we started with

That is: Define $I = \int \frac{\ln t}{t}\,dt$. Then

$$I = (\ln t)^2 - I$$

$$2I = (\ln t)^2 + C'$$

$$(C = C'/2)$$

$$I = \frac{(\ln t)^2 + C'}{2} \Rightarrow \int \frac{\ln t}{t}\,dt = \frac{1}{2}(\ln t)^2 + C$$

A.5

$$\int e^{1-t}\,dt = \int e^u \cdot \frac{du}{(-1)} = -\int e^u\,du = -e^u + C$$

$$\boxed{\begin{array}{l} u = 1-t \\ du = (-1)\,dt \end{array}}$$

$$= \underline{-e^{1-t} + C}$$

$$\left(-e^{1-t}\right)' = -1 \cdot e^{1-t} \cdot (-1) = \underline{e^{1-t}}$$

so the answer above is correct.

A.6

$$\int \frac{1}{at+b}\,dt = \int \frac{1}{u}\,\frac{du}{a} = \frac{1}{a} \cdot \int \frac{1}{u}\,du = \frac{1}{a} \cdot \ln|u| + C$$

$$\boxed{\begin{array}{l} u = at+b \\ du = a \cdot dt \end{array}}$$

$$= \underline{\frac{1}{a} \ln|at+b| + C}$$

A.7.

a) $$\int 3t\sqrt{t^2+1}\,dt = \int 3t \cdot \sqrt{u} \cdot \frac{du}{2t} = \frac{3}{2}\int u^{1/2}\,du$$

$$\boxed{\begin{array}{l} u = t^2+1 \\ du = 2t\,dt \end{array}}$$

$$= \frac{3}{2} \cdot \left(\frac{2}{3} u^{3/2}\right) + C = u^{3/2} + C = \underline{(t^2+1)\sqrt{t^2+1} + C}$$

b) $$\int \frac{t}{t^2-1}\,dt = \int \frac{t}{u} \cdot \frac{du}{2t} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$\boxed{\begin{array}{l} u = t^2-1 \\ du = 2t\,dt \end{array}}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \underline{\frac{1}{2}\ln|t^2-1| + C}$$

c) $\int 5t(t^2-1)^3\,dt \underset{\boxed{\begin{matrix}u=t^2-1\\ du=2t\,dt\end{matrix}}}{=} \int 5t\cdot u^3\cdot\frac{du}{2t}$

$= \frac{5}{2}\int u^3\,dt = \frac{5}{2}\cdot\frac{1}{4}u^4+C = \underline{\frac{5}{8}(t^2-1)^4+C}$

d) $\int \frac{2t+3}{t^2+3t+2}\,dt \underset{\boxed{\begin{matrix}u=t^2+3t+2\\ du=2t+3\end{matrix}}}{=} \int \frac{2t+3}{u}\cdot\frac{du}{2t+3}$

$= \int \frac{1}{u}\,du = \ln|u|+C = \underline{\ln|t^2+3t+2|+C}$

A.8

a) $\int t e^{t^2}\,dt \underset{\boxed{\begin{matrix}u=t^2\\ du=2t\,dt\end{matrix}}}{=} \int t\cdot e^u\cdot\frac{du}{2t} = \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u+C = \underline{\frac{1}{2}e^{t^2}+C}$

b) $\int t^3 e^{t^2}\,dt \underset{\boxed{\begin{matrix}u=t^2\\ du=2t\,dt\end{matrix}}}{=} \int t^3 e^u\cdot\frac{du}{2t} = \frac{1}{2}\int t^2 e^u\,du$

$= \frac{1}{2}\int ue^u\,du$

$= \frac{1}{2}(ue^u-e^u)+C = \underline{\frac{1}{2}t^2e^{t^2}-\frac{1}{2}e^{t^2}+C}$

c) $\int e^{\sqrt{t}}\,dt = \int e^u \cdot du \cdot 2\sqrt{t} = 2\int u e^u\,du$

$\boxed{\begin{array}{l} u=\sqrt{t} \\ du = \frac{1}{2\sqrt{t}}\,dt \end{array}}$

use $\int te^t dt$ from Section A3

$= 2(ue^u - e^u) + C = \underline{2\sqrt{t}\,e^{\sqrt{t}} - 2e^{\sqrt{t}} + C}$

d) $\int \sqrt{t}\,e^{\sqrt{t}}\,dt = \int \sqrt{t}\,e^u\,du \cdot 2\sqrt{t} = 2\int u^2 e^u\,du$

$\boxed{\begin{array}{l} u=\sqrt{t} \\ du = \frac{1}{2\sqrt{t}}\,dt \end{array}}$

Use $\int t^2e^t dt$ from Section A3

$= 2(u^2e^u - 2ue^u + 2e^u) + C$

$= \underline{2te^{\sqrt{t}} - 4\sqrt{t}\,e^{\sqrt{t}} + 4e^{\sqrt{t}} + C}$

e) $\int \frac{2e^t}{e^t + e^{-t}}\,dt = \int \frac{2u}{u + 1/u}\,\frac{du}{e^t} = \int \frac{2u}{u+1/u} \cdot \frac{du}{u}$

$\boxed{\begin{array}{l} u=e^t \\ du = e^t\,dt \end{array}}$

$= \int \frac{2u}{u^2+1}\,du = \int \frac{\cancel{2u}}{v}\,\frac{dv}{\cancel{2u}} = \int \frac{1}{v}\,dv$

$\boxed{\begin{array}{l} v=u^2+1 \\ dv = 2u\,du \end{array}}$

$= \ln|v| + C = \ln|u^2+1| + C = \underline{\ln(e^{2t}+1) + C}$

A.9

$$(t^2-3t+7) : (t-4) = \underbrace{t+1}_{\text{quotient}} + \frac{11}{t-4}$$

$$\begin{array}{l} -(t^2-4t) \\ \hline t+7 \\ -(t-4) \\ \hline 11 \end{array}$$

remainder

$\Downarrow$

$$\int \frac{t^2-3t+7}{t-4}\,dt = \int \left(t+1 + \frac{11}{t-4}\right)dt$$

$$= \underline{\tfrac{1}{2}t^2 + t + 11 \cdot \ln|t-4| + C}$$

A.10.

a) $t^2-3 : t+4 = t-4 + \frac{13}{t+4} \Rightarrow \int \frac{t^2-3}{t+4}\,dt = \int t-4 + \frac{13}{t+4}\,dt$

$$\begin{array}{l} -(t^2+4t) \\ \hline -4t-3 \\ -(-4t-16) \\ \hline \end{array}$$

$$= \underline{\tfrac{1}{2}t^2 - 4t + 13\ln|t+4| + C}$$

b) $\int \frac{t+1}{t^2+2t+4}\,dt = \int \frac{\cancel{t+1}}{u} \cdot \frac{du}{\cancel{2t+2}\,2} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$$\boxed{\begin{array}{l} u = t^2+2t+4 \\ du = (2t+2)\,dt \end{array}}$$

$$= \underline{\tfrac{1}{2}\ln|t^2+2t+4| + C}$$

c) $\int \frac{t}{t^2-4}\,dt = \int \left(\frac{1/2}{t-2} + \frac{1/2}{t+2}\right) dt$

PBD:

$t^2-4 = (t+2)(t-2)$

$\frac{t}{t^2-4} = \frac{A}{t-2} + \frac{B}{t+2} \quad \Big| \cdot (t-2)(t+2)$

$t = A\cdot(t+2) + B(t-2)$

$= \underbrace{(A+B)}_{=1}t + \underbrace{(2A-2B)}_{=0}$

$2A-2B=0 \Rightarrow A=B$

$A+B = 2A = 1 \Rightarrow A = 1/2$

$\frac{t}{t^2-4} = \frac{1/2}{t-2} + \frac{1/2}{t+2}$

$= \frac{1}{2}\ln|t-2| + \frac{1}{2}\ln|t+2| + C$

$= \frac{1}{2}\ln|(t-2)(t+2)| + C$

$= \frac{1}{2}\ln|t^2-4| + C$

(can also be solved using the substitution $\boxed{\begin{array}{l} u = t^2-4, \\ du = 2t\,dt \end{array}}$)

d) $\frac{3}{t(3-t)} = \frac{A}{t} + \frac{B}{3-t} \quad \Big| \cdot t(3-t)$

$3 = A\cdot(3-t) + Bt$

$= \underbrace{(B-A)}_{=0}t + \underbrace{3A}_{=3}$

$A=B, \quad A=1, \quad B=1$

$\frac{3}{t(3-t)} = \frac{1}{t} + \frac{1}{3-t}$

$\Rightarrow \int \frac{3}{t(3-t)}\,dt = \int \frac{1}{t} + \frac{1}{3-t}\,dt$

$= \ln|t| - \ln|3-t| + C$

$= \ln\left|\frac{t}{3-t}\right| + C$

$u = 3-t$

$du = \underline{\underline{(-1)}}\,dt$