FORELESNING 15

MET 1180 MATEMATIKK

EIVIND ERIKSEN, NOV 23, 2016

Plan:

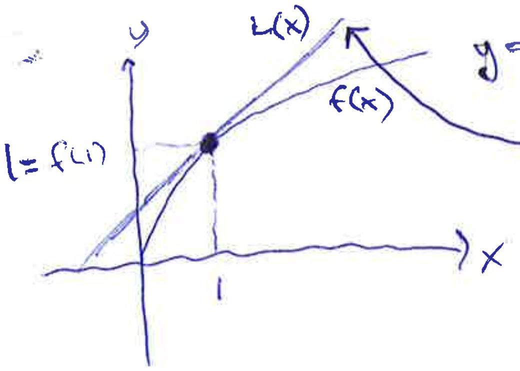
1. Lineær approksimasjon — [E] 4.10
2. Taylor-polynom og Taylor-rekker
3. Binomial formelen

Pensum: Eksamen 7. desember (MET11802)
Kap 1 – Kap 4 [E] med oppgaver.

Ekstra repetisjons-/oppsummerings forelesning: 5. desember

Repetisjon / gjennomgang av eksamens oppgave.

1. Lineær approksimasjon: Tangent linjen



$y = f(x) = \sqrt{x}$

tangenten til $y = f(x) = \sqrt{x}$ i $x = 1$
$y = L(x)$

1. $L(x) \simeq f(x)$ når x er nær 1
2. Regner ut $L(x)$:

$$y - f(1) = f'(1) \cdot (x-1)$$

Ettpunktsformel for linjen gjennom $(1, f(1))$ med stign.tall $f'(1)$

$\boxed{f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2\sqrt{x}}}$

$$y - 1 = \frac{1}{2} \cdot (x-1)$$

$$y = 1 + \frac{1}{2}(x-1)$$

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Lineær approksimasjon:

Den lineære approksimasjonen (tilnærmingen) til funksjon f i punktet $x=a$ er

$$L(x) = f(a) + f'(a)\cdot(x-a)$$

Eks: $f(x)=\sqrt{x}$ i $x=1$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(1) = \frac{1}{2}$

$$L(x) = f(1) + f'(1)\cdot(x-1)$$
$$= 1 + \frac{1}{2}\cdot(x-1) = \frac{1}{2}x + \frac{1}{2}$$

$$\sqrt{1{,}1} = f(1{,}1) \simeq L(1{,}1) = 1 + \frac{1}{2}\cdot 0{,}1 = 1{,}05$$

Tilnærming ved kvadratiske funksjoner

Eks: $f(x)=\sqrt{x}$; $x=1$

$L(x) = 1 + \frac{1}{2}(x-1)$

den beste tilnærmingen som er en rett linje.

$$Q(x) = A + B\cdot(x-1) + C\cdot(x-1)^2$$
$$= 1 + \frac{1}{2}(x-1) + \underbrace{C}_{f''(1)/2}\cdot(x-1)^2$$

← hvordan velge C optimalt?

$f(x) \simeq Q(x)$ når x er nær 1

$\Updownarrow$

$f(1) = Q(1)$
$f'(1) = Q'(1)$
$f''(1) = Q''(1)$

$Q(1) = 1 \qquad A = f(1) = 1$

$Q'(x) = B\cdot 1 + C\cdot 2\cdot(x-1) \qquad Q'(1) = B = f'(1) = 1/2$

$Q''(x) = 2C\cdot 1 = 2C \qquad 2C = f''(1) \qquad C = \frac{f''(1)}{2}$

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Kvadratisk approksimasjon

Den kvadratiske approksimasjon (tilnærming) til f i $x=a$ er

$$Q(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Eks: $f(x) = \sqrt{x}$ i $x=1$

$$L(x) = 1 + \frac{1}{2}(x-1)$$

$$Q(x) = 1 + \frac{1}{2}(x-1) + \frac{-1/4}{2}(x-1)^2$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

$$\sqrt{2} \simeq L(2) = 1 + \frac{1}{2}\cdot 1 = 1{,}5$$

$$\sqrt{2} \simeq Q(2) = 1 + \frac{1}{2}\cdot 1 - \frac{1}{8}\cdot 1^2 = 1{,}5 - 0{,}125$$

$$= 1{,}375$$

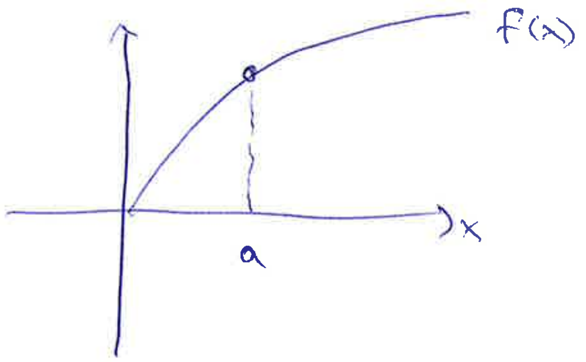
$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

$$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x\sqrt{x}}$$

$$f''(1) = -1/4$$

2 Taylor-polynom



Taylor-polynomet til f i $x=a$ av grad n:

Den beste tilnærmingen ved hjelp av n'te gradpolynomet p_n

$$p_1 = L(x) = f(a) + f'(a)\cdot(x-a)$$

$$p_2 = Q(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2$$

$$p_3 = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2 + \frac{f'''(a)}{6}\cdot(x-a)^3$$

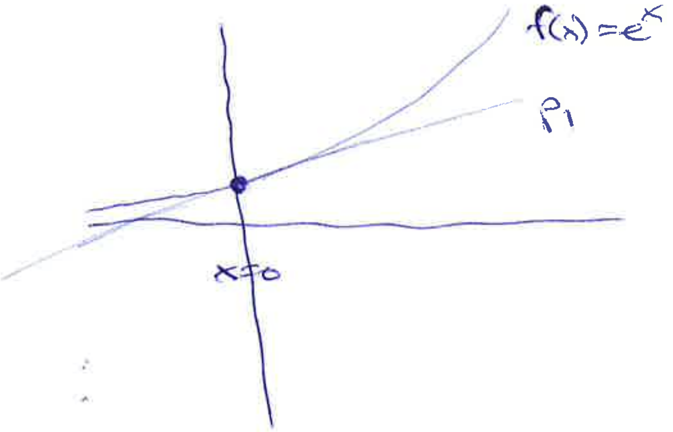
$$p_4 = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \frac{f''''(a)}{24}\cdot(x-a)^4$$

$$\boxed{\begin{aligned} p_3 &= Q(x) + C\cdot(x-a)^3 \\ p_3''' &= C\cdot\left(3\cdot(x-a)^2\right)'' \\ &= C\cdot 3\cdot\left(2\cdot(x-a)\right)' \\ &= C\cdot 3\cdot 2\cdot 1 = \underline{6C} \\ p_3''' &= 6C = f'''(a) \\ C &= \underline{\frac{f'''(a)}{6}} \end{aligned}}$$

Taylor-polynom til f av grad n i $x=a$:

$$p_n = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Eks: $f(x)=e^x$ i $x=0$. Regn ut p_1, p_2, p_3.



$$p_1 = 1 + (x-0) = 1+x$$

$$p_2 = 1 + x + \frac{1}{2}x^2$$

$$p_3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2\cdot 1}x^3$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$f(x)=e^x$	$f(0)=1$
$f'(x)=e^x$	$f'(0)=1$
$f''(x)=e^x$	$f''(0)=1$
$f'''(x)=e^x$	$f'''(0)=1$

$$p_4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$p_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

Hva skjer når $n \to \infty$?

I mange tilfeller er

$$f(x) = \lim_{n\to\infty} p_n(x)$$

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Lagrange's restleddsformel:

$R_n(x) = f(x) - p_n(x)$ feilen i tilnærmingen $f(x) \simeq p_n(x)$

Lagrange's restleddsformel:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for en c mellom a og x

Taylor-rekken til f i $x=a$:

$\lim_{n\to\infty} p_n(x)$

For "de fleste" funksjoner er $\lim_{n\to\infty} R_n(x) = 0$ (feilen blir mindre og mindre jo større n er), så er $f(x) = \lim_{n\to\infty} p_n(x)$

Eks: $f(x) = e^x$ i $x=0$

Taylor-polynom av grad $n=5$

$p_5 = p_5(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$

Taylor-rekken:

$1 + x + x^2/2 + \dots + x^n/n! + \dots = e^x$

e^x, $\ln x$ og de aller fleste andre funksjoner i dette kurset er analytiske

De funksjonene som er slik at $\lim_{n\to\infty} R_n(x) = 0$ kalles analytiske

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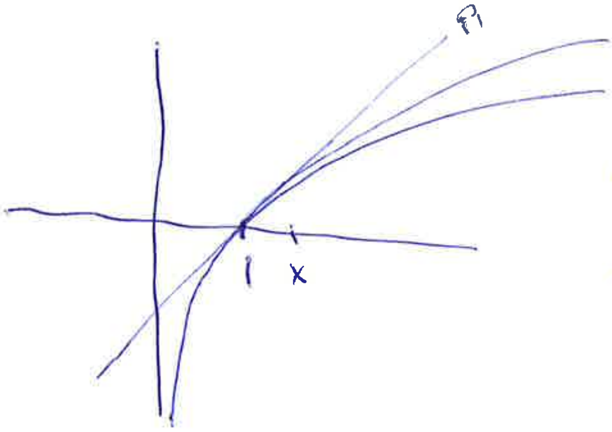
Eks: $f(x) = \ln x$; $x = 1$ Finn $p_3(x)$.



$$p_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3$$

$$p_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

	$x=1$ $(a=1)$
$f(x) = \ln x$	$0 = f(a)$
$f'(x) = \frac{1}{x}$	$1 = f'(a)$
$f''(x) = -\frac{1}{x^2}$	$-1 = f''(a)$
$f'''(x) = \frac{2}{x^3}$	$2 = f'''(a)$
$f^{(4)}(x) = \frac{-6}{x^4}$	$-6 = f^{(4)}(a)$

$$= 0 + 1\cdot(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3$$

$$= \underline{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$$

$$p_4 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$$\frac{f^{(4)}(1)}{4!} = \frac{-6}{24} = \underline{-\frac{1}{4}}$$

Taylor-rekken:

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \dots + \frac{(-1)^{n+1}}{n}(x-1)^n + \dots$$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

ln(x) er analytisk, dvs
ln(x) er lik Taylor-rekken

Lagrange's restleddsformel.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-a)^{n+1}$$ for et tall c mellom x og a

$f(x) - p_n(x)$

Eks: $f(x) = e^x$ i $x=0$

$$p_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$R_4(x) = \frac{f^{(5)}(c)}{5!}(x-0)^5 = \frac{e^c}{120}x^5$$

$R_4(x) = \frac{e^c}{120}x^5$ for en c mellom 0 og x

i $x=2$:

$$e^2 = f(2) \simeq p_4(2) = 1+2+2+8/6+16/24$$
$$= 5 + 4/3 + 2/3 = 7$$

$$e^2 \simeq 7$$

Feilen i denne tilnærmingen:

$$R_4(2) = \frac{e^c}{120} \cdot 2^5 = \frac{32}{120}e^c$$ der $c \in (0,2)$

$$\leq \frac{32}{120}e^2$$

$$R_5(2) = \frac{e^c}{720} \cdot 64 \leq \frac{64}{720}e^2$$

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3 Binomial-formelen : Newton's binomial formel

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

$$n! = n\cdot(n-1)(n-2)\cdots 2\cdot 1$$

(n fakultet) $\boxed{0! = 1}$

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!\,(n-r)!} = \frac{n\cdot(n-1)(n-2)\cdots \cancel{4}\cdot 3\cdot 2\cdot 1}{\underbrace{r\cdot(r-1)\cdots 2\cdot 1}_{r!}\cdot\underbrace{(n-r)(n-r-1)\cdots 2\cdot 1}_{(n-r)!}}$$

(binomial koeff. n over r)

$$= \frac{n(n-1)\cdots(n-r+1)}{r!}$$

= antall måter å trekke r obj. fra en mengde med n obj.

Eks:

$$\binom{3}{1} = \frac{3\cdot\cancel{2}\cdot\cancel{1}}{(1)\,\cancel{(2\cdot 1)}} = \underline{3}$$

$$\binom{3}{2} = \frac{3\cdot 2\cdot\cancel{1}}{2\cdot 1\cdot\cancel{1}} = \underline{3}$$

$$\binom{3}{3} = \frac{\cancel{3\cdot 2\cdot 1}}{\cancel{3\cdot 2\cdot 1}\cdot 1} = \underline{1}$$

$$\binom{3}{0} = \frac{3\cdot 2\cdot 1}{0!\cdot 3\cdot 2\cdot 1} = \underline{1}$$

$\binom{4}{0} = 1$ $\binom{4}{1} = 4$ $\binom{4}{2} = 6$

$\binom{4}{3} = 4$ $\binom{4}{4} = 1$

Pascal's trekant

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

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Eks: $f(x) = (x+1)^n$ i $x=0$

	$x=0$	
$f(x)=(x+1)^n$	1	$= f(0)$
$f'(x) = n\cdot(x+1)^{n-1}$	n	$= f'(0)$
$f''(x) = n\cdot(n-1)\cdot(x+1)^{n-2}$	$n(n-1)$	$= f''(0)$
$f'''(x) = n\cdot(n-1)(n-2)\cdot(x+1)^{n-3}$	$n(n-1)(n-2)$	$= f'''(0)$
$f^{(4)}(x) = n\cdot(n-1)(n-2)\cdot(n-3)\cdot(x+1)^{n-4}$	$n(n-1)(n-2)(n-3)$	

	$x=0$
$f^{(5)}(x)$	
$\vdots$	
$f^{(n)}(x) = n!$	$n!$
$f^{(n+1)}(x) = 0$	0

$$p_1 = 1 + n\cdot x$$

$$p_2 = 1 + nx + \frac{n(n-1)}{2}x^2$$

$$p_3 = \underset{\binom{n}{0}}{\underset{\uparrow}{1}} + \underset{\binom{n}{1}}{\underset{\uparrow}{nx}} + \underset{\binom{n}{2}}{\underset{\uparrow}{\frac{n(n-1)}{2}}}x^2 + \underset{\binom{n}{3}}{\underset{\uparrow}{\frac{n(n-1)(n-2)}{3!}}}x^3$$

$$f(x) = (x+1)^n = p_n(x) = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots + \frac{n!}{n!}x^n$$

$$(x+1)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$$

Eks: $(x+1)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

$(x+1)^3 = 1 + 3x + 3x^2 + x^3$

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Binomial formel:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots$$

$$\ldots + \binom{n}{n-1} a b^{n-1} + b^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Eks: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$