BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

FORELESNING 24

METI180 MATEMATIKK

EIVIND ERIKSEN MAR 07, 2017

Plan:

1. Inverse matriser
2. Eksamensoppgave 12/2015, Opg 3

Pensum: [EJ] 6.6

Husk: Innleveringsoppgave (frist tir 14/03)

Forelesning onsdag 08/03 (kl. 08-11)

(1) Inverse matriser

A kvadratisk $n \times n$-matrise

Defn: B kalles en invers matrise for A hvis

$$A \cdot B = I_n \quad \text{og} \quad \underset{A^{-1}}{B} \cdot A = I_n$$

I såfall er $\underset{A^{-1}}{B}$ entydig og skrives $A^{-1} = B$.

$n=2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}:$$

$$\begin{cases} |A| \neq 0: & A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ |A| = 0: & \text{ingen invers matrise } A^{-1} \end{cases}$$

Eks:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$|A| = 2 \cdot 2 - 1 \cdot 1 = \underline{3} \neq 0$$

$$A^{-1} = \frac{1}{3} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}}}$$

$$\begin{aligned} 2x + y &= 7 \\ x + 2y &= 5 \end{aligned} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}}$$

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$n > 2$: A $n \times n$-matrise

den adjungerte matrisen til A.

a) Formel for A^{-1}:

$$\begin{cases} |A| \neq 0: & A^{-1} = \frac{1}{|A|} \cdot \mathrm{adj}(A) \\ |A| = 0: & A^{-1} \text{ finnes ikke} \end{cases}$$

$\mathrm{adj}(A) = C^T$, der $C = (c_{ij})$ er kofaktormatrisen

Eks: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$$|A| = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1 \cdot 6 - 1 \cdot 5 + 1 \cdot 1 = \underline{\underline{2}} \neq 0$$

$$A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$C_{11} = 6$ $\quad C_{12} = -5$ $\quad C_{13} = 1$

$C_{21} = -6$ $\quad C_{22} = 8$ $\quad C_{23} = -2$

$C_{31} = 2$ $\quad C_{32} = -3$ $\quad C_{33} = 1$

$$A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T = \underline{\underline{\frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

$$= \underline{\underline{\begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}}}$$

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Eks:

$$\begin{aligned} x+y+z &= a \\ x+2y+4z &= a \\ x+3y+9z &= 2a \end{aligned} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a \\ 2a \end{pmatrix}$$

$$\underbrace{\cancel{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}^{-1} \cdot \underbrace{\begin{pmatrix} a \\ a \\ 2a \end{pmatrix}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ a \\ 2a \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 4a \\ -3a \\ a \end{pmatrix} = \begin{pmatrix} 2a \\ -3a/2 \\ a/2 \end{pmatrix}$$

Én løsning for hver parameterverdi a:

$x = 2a$

$y = -3a/2$

$z = a/2$

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b) A^{-1} vha Gauss

hvis vi får $\left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & \end{array}\right)$ betyr det at A^{-1} ikke finnes

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \to \dots \to (I \mid \overset{A^{-1}}{*})$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right) \begin{matrix} \\ \\ -2 \end{matrix} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \cdot \tfrac{1}{2}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \begin{matrix} \\ -3 \\ -1 \end{matrix}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 1 & -1/2 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) -1$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) = (I \mid A^{-1})$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

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Viktige regneregler for matriser:

$$A \cdot B \neq B \cdot A$$

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

$$(A \cdot B)^{T} = B^{T} \cdot A^{T}$$

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

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Eks. 12/2015, Oppgave 3: (MET11803)

$$A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}$$

a) $|A| = a \cdot \begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 \\ 2 & a \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & a \\ 2 & 2 \end{vmatrix}$

$= a(a^2-4) - 2(2a-4) + 2(4-2a)$

$= a^3 - 4a - 4 \cdot a + 8 + 8 - 4a$

$= \underline{\underline{a^3 - 12a + 16}}$ full utregning i a)

$-4 \cdot (a-2)$

(alt.)

$= a \cdot (a-2)(a+2) - 4 \cdot (a-2) + 4 \cdot (2-a)$

$= (a-2) \cdot [a(a+2) - 4 - 4]$

$= (a-2)(a^2+2a-8)$

$= (a-2)(a-2)(a+4)$

$= \underline{\underline{(a-2)^2(a+4)}}$

(trenger faktorisering i oppgave c))

$a^2+2a-8=0$

$a = \frac{-2 \pm \sqrt{4+32}}{2} = \frac{-2 \pm 6}{2} = 2, -4$

b) $\underline{a=0}$: Finn A^{-1}. $|A| = \underline{16} \neq 0 \Rightarrow A^{-1}$ eksisterer.

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{16} \cdot \begin{pmatrix} -4 & 4 & 4 \\ 4 & -4 & 4 \\ 4 & 4 & -4 \end{pmatrix}^T = \frac{1}{16} \cdot \begin{pmatrix} -4 & 4 & 4 \\ 4 & -4 & 4 \\ 4 & 4 & -4 \end{pmatrix}$$

symm. symm.

$$= \underline{\underline{\begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 \end{pmatrix}}}$$

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b)

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$c_{11} = +(0 \cdot 0 - 2 \cdot 2) = \underline{-4}$$

$$c_{12} = -(2 \cdot 0 - 2 \cdot 2) = -(-4) = \underline{4}$$

$\vdots$

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c) $|A| = a^3 - 12a + 16$

$= \underline{(a-2)^2 \cdot (a+4)}$

$|A| = 0$: $a=2,\ a=-4$ $\longrightarrow$ { ingen løsn. eller uendelig mange løsn.

$|A| \neq 0$: $a \neq 2, a \neq -4$ $\longrightarrow$ én løsning

Uendelig mange løsn for $\underline{\underline{a=2,\ a=-4}}$

$\underline{a=2}$:

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{array}\right) \begin{array}{l} \\ -1 \\ -1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

uendelig mange løsn.
(y, z frie)

$\underline{2x} + 2y + 2z = 2$
$y = \text{fri}$
$z = \text{fri}$

$$\boxed{\begin{array}{l} x = 1 - y - z \\ \left.\begin{array}{l} y = y \\ z = z \end{array}\right\} \text{fri} \end{array}}$$

$\underline{a=-4}$:

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 2 \\ 2 & -4 & 2 & -4 \\ 2 & 2 & -4 & 2 \end{array}\right) \begin{array}{l} \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \end{array}$$

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 3 & -3 & 3 \end{array}\right) \begin{array}{l} \\ \\ 1 \end{array}$$

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

uendelig mange løsn.
(z fri)

$\underline{-4x} + 2y + 2z = 2$
$\underline{-3y} + 3z = -3$
z fri

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$\underline{a=-4}$:

$$\boxed{\begin{aligned} x &= z \\ y &= z+1 \\ z &= z \quad (\text{fri}) \end{aligned}} \longleftarrow \quad \frac{-3y}{-3} = \frac{-3z-3}{-3}$$

$$y = \underline{z+1}$$

$$-4x + 2(z+1) + 2z = 2$$

$$-4x + 4z + \cancel{2} = \cancel{2}$$

$$\frac{-4x}{-4} = \frac{-4z}{-4}$$

$$\underline{x = z}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{\underline{\begin{pmatrix} z \\ z+1 \\ z \end{pmatrix}}}$$

d) $A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 2 \\ a \\ 2 \end{pmatrix}$ $\quad a \neq 2, -4$

$$x = \frac{|A_1(b)|}{|A|} = \frac{\begin{vmatrix} 2 & 2 & 2 \\ a & a & 2 \\ 2 & 2 & a \end{vmatrix}}{(a-2)^2(a+4)} = \frac{2\cdot 0 - 2\cdot 0 + a \cdot 0}{(a-2)^2(a+4)} = \underline{\underline{0}}$$

$$y = \frac{|A_2(b)|}{|A|} = \frac{\begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix}}{(a-2)(a+4)} = \frac{|A|}{|A|} = \underline{\underline{1}}$$

$$z = \frac{|A_3(b)|}{|A|} = \frac{\begin{vmatrix} a & 2 & 2 \\ 2 & a & a \\ 2 & 2 & 2 \end{vmatrix}}{|A|} = \frac{0}{|A|} = \underline{\underline{0}}$$