FORELESNING 18

MET 1180

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MATEMATIKK

BI

Plan:

1. Substitusjon
2. Delvis integrasjon
3. Delbrøksoppspaltning

Pensum:

[EJ] 5.3–5.5

Husk:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

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1. Substitusjon:

Metode:

i) Velg $\boxed{u = u(x)}$ ← uttrykk i x

ii) Bruk likningen $\boxed{du = u' \cdot dx}$ $\leadsto dx = \frac{1}{u'} \cdot du$

til å skrive om integralet til et integral i u.

iii) Løs integralet i u, og skriv om svaret til et uttrykk i x.

$$\int \frac{x}{x^2+1}\,dx$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

$$dx = \frac{1}{2x} \cdot du$$

$$= \int \frac{\cancel{x}}{u} \cdot \frac{1}{2\cancel{x}} \cdot du$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2} \cdot \ln|u| + C$$

$$= \underline{\underline{\frac{1}{2}\ln(x^2+1) + C}}$$

Opps 5.3.3

a) $\displaystyle\int \frac{3x}{x^2+1}\,dx = \int \frac{3\cancel{x}}{u}\cdot\frac{1}{2\cancel{x}}\,du = \frac{3}{2}\cdot\int\frac{1}{u}\,du$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

$= \frac{3}{2}\ln|u| + C$

$= \underline{\underline{\frac{3}{2}\ln(x^2+1)+C}}$

b) $\displaystyle\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}}\cdot\frac{du}{\cancel{1/x}} = \int u\,du$

$\ln x \cdot \frac{1}{x}$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$= \frac{1}{2}u^2 + C$

$= \underline{\underline{\frac{1}{2}(\ln x)^2 + C}}$

Husk: $\ln(x^2) = 2\ln x$
$(\ln x)^2 \neq 2\ln x$

c) $\displaystyle\int \frac{x}{1-x}\,dx = \int \frac{x}{u}\cdot\frac{du}{-1} = -\int \frac{1-u}{u}\,du$

$$\boxed{\begin{aligned} u &= 1-x \\ du &= -1\cdot dx \end{aligned}}$$

$\rightarrow x = 1-u$

Husk: $\frac{x}{1-x} \neq x-1$

$= -\displaystyle\int \frac{1}{u}-1\,du = -\left(\ln|u| - u\right) + C$

$= -\ln|u| + u + C = \underline{\underline{-\ln|1-x| + 1 - x + C}}$

$= \underline{\underline{-\ln|1-x| - x + C}}$

d) $\int x^3 e^{x^2}\,dx \neq \int x^{3} \cdot e^{u} \cdot \frac{1}{2x}\,du$

$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x \cdot dx \end{aligned}}$

$$\int x^3 e^{x^2}\,dx = \int x^{\cancel{3}2} e^{u} \cdot \frac{1}{2\cancel{x}}\,du = \frac{1}{2}\int x^2 e^{u}\,du$$

$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x \cdot dx \end{aligned}}$

$$= \frac{1}{2}\int u e^{u}\,du$$

delvis int.

den enkleste måten å løse dette integralet på, er ved delvis integrasjon

$$= \frac{1}{2}\left(ue^{u} - e^{u}\right) + C = \frac{1}{2}ue^{u} - \frac{1}{2}e^{u} + C$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$$

e) $\int \frac{\sqrt{x}}{1-x}\,dx = \int \frac{\sqrt{1-u}}{u}\,\frac{du}{-1} =$

$\boxed{\begin{aligned} u &= 1-x \\ du &= -1 \cdot dx \end{aligned}}$

$$= \int \frac{u}{1-x}\,\frac{du}{\frac{1}{2\sqrt{x}}} = \int \frac{u \cdot 2\sqrt{x}}{(1-x)}\,du =$$

$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$

$$= \int \frac{2u \cdot u}{1-u^2}\,du = \int \frac{2u^2}{1-u^2}\,du$$

(fortsetter etter delbrøksoppspaltn.)

(2) Delvis integrasjon

(Hoved-sakelos)

Metode for å integrere et produkt.

$$\boxed{\int u' \cdot v \, dx = uv - \int u v' \, dx}$$

Eks:

$$\int \overset{u'}{x} \cdot \overset{v}{\ln x} \, dx = \frac{1}{2}x^2 \cdot \ln x - \int \overset{u}{\frac{1}{2}x^2} \cdot \overset{v'}{\frac{1}{x}} \, dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$$

$$\int \overset{u'}{\ln x} \cdot \overset{v}{x} \, dx =$$

$$\boxed{\begin{array}{ll} u = ? & v = x \\ u' = \ln x & v' = 1 \end{array}}$$

Hvorfor?

$$(uv)' = u'v + uv'$$

$$\int (uv)'\,dx = \int u'v + uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\boxed{\int u'v\,dx = uv - \int uv'\,dx}$$

$$\int uv'\,dx = uv - \int u'v\,dx \qquad \text{Alt.}$$

Eks:

$$\int x \cdot e^x\,dx = xe^x - \int e^x \cdot 1\,dx$$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}} \qquad \cancel{\begin{array}{ll} u = \tfrac{1}{2}x^2 & v = e^x \\ u' = x & v' = e^x \end{array}}$$

$$= xe^x - \int e^x\,dx = \underline{\underline{xe^x - e^x + C}}$$

Eks:

$$\int \ln x\,dx = \int \underset{u'}{1} \cdot \underset{v}{\ln x}\,dx = x \cdot \ln x - \int x \cdot \frac{1}{x}\,dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int 1\,dx = \underline{\underline{x \cdot \ln x - x + C}}$$

$$\boxed{\int \ln x\,dx = x \cdot \ln x - x + C}$$

$$(x\ln x - x + C)'$$
$$= 1 \cdot \ln x + x \cdot \frac{1}{x} - 1$$
$$= \ln x + 1 - 1 = \underline{\ln x}$$

③ Delbrøksoppspalting.

$$\int \frac{2}{1-x^2}\,dx = ?$$

$$\neq 2\ln|1-x^2|+C \leftarrow$$ vanlig feil

$$\int \frac{2}{1-x^2}\,dx = \int \frac{\cancel{2}}{u}\cdot\frac{du}{-\cancel{2}x}$$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$= \int \frac{1}{u\cdot(-x)}\,du = \dots$$

fører ikke frem

Metode for å integrere rasjonale funksjoner

$$\int f(x)\,dx = \int \frac{p(x)}{q(x)}\,dx$$

$$\int \frac{1}{x}\,dx = \ln|x|+C$$

$$\int \frac{1}{1-x}\,dx = \int \frac{1}{u}\cdot\frac{du}{-1}$$

$$\boxed{\begin{aligned} u &= 1-x \\ du &= -dx \end{aligned}}$$

$$= -\int \frac{1}{u}\,du = -\ln|u|+C$$

$$= \underline{-\ln|1-x|+C}$$

Formel:

$$\int \frac{A}{ax+c}\,dx = \frac{A}{a}\ln|ax+b| + C$$

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Delbrøksoppspalting:

$$\frac{2}{1-x^2} = \frac{2}{(1-x)\cdot(1+x)} = \frac{A}{1-x} + \frac{B}{1+x} \leftarrow$$

prøver å finne konstanter A og B slik at dette stemmer

$$\frac{2}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x} \quad \Big| \cdot(1-x)(1+x)$$

$$2 = \frac{A\cdot\cancel{(1-x)}(1+x)}{\cancel{1-x}} + \frac{B\cdot(1-x)\cancel{(1+x)}}{\cancel{1+x}}$$

$$2 = A\cdot(1+x) + B(1-x)$$

$$2 = A + Ax + B - Bx = \underbrace{(A+B)}_{=2} + \underbrace{(A-B)}_{=0}x$$

$A+B=2$
$A-B=0$

$A=B$
$2A=2$

$A=1$
$B=1$

Konkl.: $\frac{2}{1-x^2} = \frac{2}{(1-x)(1+x)} = \frac{1}{1-x} + \frac{1}{1+x}$

delbrøksoppspaltning

$$\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1-x} + \frac{1}{1+x}\,dx = \int \frac{1}{1-x}\,dx + \int \frac{1}{1+x}\,dx$$

$$= \underline{\underline{-\ln|1-x| + \ln|1+x| + C}}$$

$\ln a + \ln b = \ln a \cdot b$

$\ln a - \ln b = \ln \frac{a}{b}$

$$= \ln|1+x| - \ln|1-x| + C = \underline{\underline{\ln\left|\frac{1+x}{1-x}\right| + C}}$$

I detalj: $\int \frac{1}{1-x}\,dx = \int \frac{1}{u}\cdot\frac{du}{-1} = \frac{1}{-1}\int \frac{1}{u}\,du = -\ln|u| + C$

$u = 1-x$

$du = -1\cdot dx$

$$= \underline{-\ln|1-x| + C}$$

$\int \frac{1}{1+x}\,dx = \int \frac{1}{u}\,\frac{du}{1} = \frac{1}{1}\cdot\int \frac{1}{u}\,du = \underline{\ln|1+x| + C}$

$u = 1+x$

$du = 1\cdot dx$

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$$

Oppsummering av metode: Integrasjon av rasjonale funksjoner

i) Bruk polynomdiv. hvis grad teller $\geq$ grad nevner.

ii) Hvis nevner har grad én: $\int \frac{A}{ax+b}\,dx = \underline{\frac{A}{a}\ln|ax+b| + C}$ ved substitusjon $\underline{u = ax+b}$.

Hvis nevner har grad ≥ 2: delbrøksoppspaltning

$$\int \frac{\sqrt{x}}{1-x}\,dx = \int \frac{2u^2}{1-u^2}\,du$$

$u=\sqrt{x}$

5.3.3 e) fortsatt

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Polynomdivisjon

Bruk poly. div. hvis grad. teller er like stor eller større enn grad. nevner.

$$2u^2 : (-u^2+1) = -2$$
$$\frac{-(2u^2-2)}{2}$$

$$\frac{2u^2}{1-u^2} = \underline{-2} + \frac{2}{1-u^2}$$

$$= \int -2 + \frac{2}{1-u^2}\,du = \int -2\,du + \int \frac{2}{1-u^2}\,du$$

$$= -2u + \int \frac{2}{1-u^2}\,du = -2u + \ln\left|\frac{1+u}{1-u}\right| + C$$

delbrøks-oppspalting

$$= -2\sqrt{x} + \ln\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| + C$$