FORELESNING 25 — MET1180 — BI

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Plan:

1. Inverse matriser, transponering
2. Regneregler for matriser
3. Eksamen 12/2015 Opps. 3 ← Neste gang

Pensum:

[E] 6.6

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① Inverse matriser

A
$n \times n$-matrise
(kvadratisk)

Defn: A^{-1} (den inverse matrise) "$\frac{1}{A}$"
er en matrise slik at

$$A^{-1} \cdot A = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

og

$$A \cdot A^{-1} = I$$

Resultat: i) A^{-1} fins (A inverterbar / invertibel)

$$\Updownarrow$$

$$|A| \neq 0$$
$$(\det(A) \neq 0)$$

ii) Hvis $|A| \neq 0$, så er

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}^T$$

(A^{-1} er entydig)

Eks:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & -2 \end{pmatrix}$$

$$|A| = -2 \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} = -2 \cdot 1 + 1 \cdot 2 = \underline{0}$$

A^{-1} fins ikke.

Eks:

$$A = \begin{pmatrix} \boxed{1} & 2 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & -3 \end{pmatrix}$$

$$|A| = -2 \cdot (-6+5) + 1 \cdot (-3+4) = -2 \cdot (-1) + 1 \cdot 1 = \underline{3} \neq 0$$

$$A^{-1} = \frac{1}{3} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{3} \begin{pmatrix} -3 & 6 & 6 \\ 1 & 1 & 3 \\ 1 & -2 & -3 \end{pmatrix}^T$$

$$= \frac{1}{3} \cdot \begin{pmatrix} -3 & 1 & 1 \\ 6 & 1 & -2 \\ 6 & 3 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 1/3 & 1/3 \\ 2 & 1/3 & -2/3 \\ 2 & 1 & -1 \end{pmatrix}$$

$C_{11} = + \begin{vmatrix} 1 & 0 \\ 5 & -3 \end{vmatrix} = -3$ $\quad C_{12} = - \begin{vmatrix} 2 & 0 \\ 4 & -3 \end{vmatrix} = 6$ $\quad C_{13} = 6$

$C_{21} = 1$ $\quad C_{22} = 1$ $\quad C_{23} = 3$

$C_{31} = 1$ $\quad C_{32} = -2$ $\quad C_{33} = -3$

BI

Lineært system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

der antall variabler = antall likninger

$A \cdot \underline{x} = \underline{b}$

(A $n \times n$-matrise)

$|A| \neq 0$:

én løsning

A^{-1} fins

$\Downarrow$

$A\underline{x} = \underline{b}$

$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

$|A| = 0$:

ingen løsn. eller uendelig mange løsn.

(hvilken kommer an på $\underline{b}$)

Eks:

$$\begin{aligned} x + 2y - z &= 4 \\ 2x + y \quad &= 1 \\ 4x + 5y - 2z &= 9 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & -2 \end{pmatrix}$$

$|A| = 0$

$\Downarrow$

ingen løsn. eller uendelig mange løsn

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 1 & 0 & 1 \\ 4 & 5 & -2 & 9 \end{array}\right)$$

Må bruke Gauss og regne ut.

Eks:

$$\begin{aligned} x + 2y - z &= 4 \\ 2x + y \quad &= 1 \\ 4x + 5y - 3z &= 9 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & -3 \end{pmatrix}$$

$|A| = 3 \Rightarrow$ én løsning

$-3 \cdot 4 + 1 \cdot 1 + 1 \cdot 9$
$= -12 + 1 + 9 = -2$

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \cdot \underline{b} = \frac{1}{3}\begin{pmatrix} -3 & 1 & 1 \\ 6 & 1 & -2 \\ 6 & 3 & -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$$

$$= \frac{1}{3} \cdot \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 7/3 \\ 0 \end{pmatrix} \begin{matrix} \leadsto x \\ \leadsto y \\ \leadsto z \end{matrix}$$

Alternativ metode for å finne A^{-1}:

Eks: $A = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}$

$$(A \mid I) = \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 3 & 10 & 0 & 1 \end{array}\right) \; -3$$

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \; 2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & -2 & -3 & 1 \end{array}\right) \quad :(-2) \qquad \cdot\left(\frac{1}{-2}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right) = (I \mid A^{-1})$$

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3/2 & -1/2 \end{pmatrix}$$

Metode: For å finne A^{-1}

1. Skriv ned $(A \mid I)$.
2. Bruk Gauss til å finne redusert trappeform til $(A \mid I)$.

To muligheter:

for eks. $\left(\begin{array}{cc|cc} 1 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right)$

$(I \mid A^{-1})$ eller $(\neq I \mid *)$

A^{-1} fins $\qquad$ A^{-1} fins ikke

Eks: $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & -3 \end{pmatrix}$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & -3 & 0 & 0 & 1 \end{array}\right) \cdot 2 \;\; \cdot 4$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & -3 & 1 & -4 & 0 & 1 \end{array}\right) \cdot -1$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right) \;\; 2 \;\; -1$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 1 & -1 \\ 0 & -3 & 0 & -6 & -1 & 2 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right) \begin{array}{l} \\ :-3 \\ :-1 \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 1 & -1 \\ 0 & 1 & 0 & +2 & +1/3 & -2/3 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right) -2$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\cancel{7} & \cancel{5}/3 & +\cancel{4}/3 \\ 0 & 1 & 0 & +2 & +1/3 & -2/3 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right) = (I \,|\, A^{-1})$$

Konkl: A^{-1} fins, og $A^{-1} = \begin{pmatrix} -1 & 1/3 & 1/3 \\ 2 & 1/3 & -2/3 \\ 2 & 1 & -1 \end{pmatrix}$ //

Regneregler for inverser:

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

Eks: $(AB)\,\underline{x} = \underline{b}$

$$\cancel{A^{-1}} \cdot \cancel{A}B\,\underline{x} = A^{-1}\,\underline{b}$$

$$B\underline{x} = A^{-1}\,\underline{b}$$

$$\cancel{B^{-1}} \cdot \cancel{B}\,\underline{x} = B^{-1} \cdot A^{-1}\,\underline{b}$$

$$\underline{x} = (B^{-1} \cdot A^{-1})\,\underline{b} \longleftarrow (AB)^{-1} = B^{-1}A^{-1}$$

Transponering: $\underset{m \times n\text{-matrise}}{A} \rightsquigarrow \underset{n \times m\text{-matrise}}{A^T} =$

Eks: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Defn: A kalles symmetrisk hvis $A^T = A$,

dvs hvis $a_{ij} = a_{ji}$ for alle i, j.

Eks: $A = \begin{pmatrix} 1 & 2 & 7 \\ 2 & 3 & -1 \\ 7 & -1 & 4 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 2 & 7 \\ 2 & 3 & -1 \\ 7 & -1 & 4 \end{pmatrix}$

Symmetrisk

2 Regneregler

BI

1. $|A \cdot B| = |A| \cdot |B|$
2. $(AB)^{-1} = B^{-1} \cdot A^{-1}$
3. $(AB)^T = B^T \cdot A^T$

Husk:

$AB \neq BA$

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3 Opps. 6.6.5:

$$A = \begin{pmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{pmatrix}$$

For hvilke s er A invertbar (A^{-1} fins)?

A^{-1} fins $\Leftrightarrow$ $|A| \neq 0$:

$$|A| = 1 \cdot (s-1) - 1 \cdot (1-s) + s \cdot (1-s^2)$$

$$= s-1-1+s+s(1-s^2)$$

$$= s-1-1+s+s-s^3 = -s^3+3s-2$$

$$= \underbrace{s(1-s)(1+s)} + \underbrace{2s-2}_{2(s-1)} = (s-1) \cdot [-s(1+s)+2]$$

$$= (s-1) \cdot (-s^2-s+2)$$

$|A|=0$: $-s^3+3s-2=0$

$(s-1) \cdot (-s^2-s+2) = 0$

$s=1$ eller $s = \frac{1 \pm \sqrt{1+8}}{-2}$

$= \frac{1 \pm 3}{-2}$

$s=-2$, $s=1$

Mulige heltallsrøtter:

$s = \pm 1, \pm 2$

$s=1$: $-1+3-2=0$

$(-s^3+3s-2) : (s-1) = -s^2-s+2$

$-(-s^3+s^2)$

$= 2s-2 \quad -s^2+3s-2 \quad (-s^2+s)$

Konkl: $|A| = 0$ hvis $s = 1$, $s = -2$

$|A| \neq 0$ hvis $s \neq 1$, $s \neq -2$

A invertibel for $s \neq 1, -2$

$s \neq 1, -2$:

$$A^{-1} = \frac{1}{-s^3+3s-2} \cdot \begin{pmatrix} s-1 & s-1 & 1-s^2 \\ s-1 & 1-s^2 & s-1 \\ 1-s^2 & s-1 & s-1 \end{pmatrix}^T$$

$$A = \begin{pmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{-s^3+3s-2} \begin{pmatrix} s-1 & s-1 & 1-s^2 \\ s-1 & 1-s^2 & s-1 \\ 1-s^2 & s-1 & s-1 \end{pmatrix}$$

Merk: Hvis A er symmetrisk, så blir også kofaktormatrisen symmetrisk.

Opps. 6.6.6.

$$A\underline{x} = \underline{b} \qquad A = \begin{pmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} x + y + sz &= 1 \\ x + sy + z &= 2 \\ sx + y + z &= 1 \end{aligned}}$$

a) $|A| = -s^3 + 3s - 2$ (forrige opps.)

$|A| = 0$: $s = 1, -2$ — " —

b) For hvilke s er systemet inkonsistent?

$A\underline{x} = \underline{b}$ inkonsistent (ingen løsn.) $\Rightarrow$ $|A| = 0$ $(s = 1,\ s = -2)$

$\underline{s = 1:}$

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

ingen løsn.

$\underline{s = -2:}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 2 \\ -2 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 1 \\ 0 & 3 & -3 & 3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

ingen løsn.

Inkonsistent for $\underline{s = 1,\ s = -2}$

c) Løs det lineære systemet i alle tilfeller hvor det er konsistent.

$s \neq 1, -2$: én løsn. $(|A| \neq 0)$

$s = 1, -2$: ingen løsn. $(|A| = 0)$

$\underline{s \neq 1, -2}$: én løsn. $\underline{x} = \ldots$

A) $\underline{\text{Gauss}}$:
$$\left(\begin{array}{ccc|c} 1 & 1 & s & 1 \\ 1 & s & 1 & 2 \\ s & 1 & 1 & 1 \end{array}\right) \xrightarrow{-1,\ -s} \left(\begin{array}{ccc|c} 1 & 1 & s & 1 \\ 0 & s-1 & 1-s & 1 \\ 0 & 1-s & 1-s^2 & 1-s \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & s & 1 \\ 0 & s-1 & 1-s & 1 \\ 0 & 0 & 2-s-s^2 & 2-s \end{array}\right)$$ én løsning.

$$2 - s - s^2 = 0$$
$$-s^2 - s + 2 = 0$$
$$s = \frac{1 \pm \sqrt{1+8}}{-2} = \frac{1 \pm 3}{-2}$$
$$s = -2, \quad s = 1$$

$$\begin{aligned} x + y + sz &= 1 \\ (s-1)y + (1-s)z &= 1 \\ (2-s-s^2)z &= 2-s \end{aligned}$$

$$z = \frac{2-s}{2-s-s^2}$$

$$y = \frac{1-(1-s)z}{s-1} = \frac{1-(1-s)\cdot\frac{2-s}{2-s-s^2}}{s-1}$$

$$= \ldots$$

$$x = 1 - y - sz = \ldots$$

b) Bruker Cramer's regel når vi har parametre:

$$A = \begin{pmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad |A| = -s^3+3s-2$$

$$x = \frac{|A_1(\underline{b})|}{|A|} = \frac{-s^2+3s-2}{-s^3+3s-2} = \frac{s-2}{(s-1)(s+2)}$$

$$y = \frac{|A_2(\underline{b})|}{|A|} = \frac{-2s^2+2s}{-s^3+3s-2} = \frac{2s}{(s-1)(s+2)}$$

$$z = \frac{|A_3(\underline{b})|}{|A|} = \frac{-s^2+3s-2}{s^3+3s-2} = \frac{s-2}{(s-1)(s+2)}$$

etter faktorisering og forkorting

$$|A_1(\underline{b})| = \begin{vmatrix} 1 & 1 & s \\ 2 & s & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\cdot(s-1) - 1\cdot 1 + s\cdot(2-s) = s-1-1+2s-s^2 = -s^2+3s-2$$

$$|A_2(\underline{b})| = \begin{vmatrix} 1 & 1 & s \\ 1 & 2 & 1 \\ s & 1 & 1 \end{vmatrix} = 1\cdot 1 - 1(1-s) + s\cdot(1-2s) = -2s^2+2s$$

$$|A_3(\underline{b})| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & s & 2 \\ s & 1 & 1 \end{vmatrix} = 1\cdot(s-2) - 1\cdot(1-2s) + 1\cdot(1-s^2) = -s^2+3s-2$$

c) Bruk A^{-1}:

$$s \neq 1, -2: \quad A^{-1} = \frac{1}{-s^3+3s-2}\begin{pmatrix} s-1 & s-1 & 1-s^2 \\ s-1 & 1-s^2 & s-1 \\ 1-s^2 & s-1 & s-1 \end{pmatrix}$$

(fra 6.6.5)

$$\Downarrow$$

$$\underline{x} = A^{-1}\cdot\underline{b} = \frac{1}{-s^3+3s-2}\cdot\begin{pmatrix} s-1 & s-1 & 1-s^2 \\ s-1 & 1-s^2 & s-1 \\ 1-s^2 & s-1 & s-1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \ldots$$