Plan:

1. Repetisjon Kap. 7
2. Eksamensoppgaver

Repetisjonsforelesning: 24 mai. Eksamen 05/2017

Kontortid: Alle dager fram til eksamen.

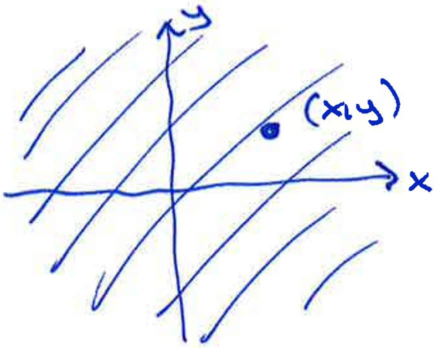
Eksamenssett III: Legges ut i løpet av den neste uken.

Se forelesningsplan for oppdateringer.

# ① Repetisjon Kap. 7: Funksjoner i flere variable (to)

## max/min uten bibetingelser

Eks: min $f(x,y) = x^3 - 3xy + y^3$




Tillatte pkt:

$\mathbb{R}^2$ = alle pkt $(x,y)$ i planet, dvs

alle pkt. er indre, ingen randpkt.

Metode:

$$\left.\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}\right\} \text{stasjonære pkt} = \text{kandidater for max/min}$$

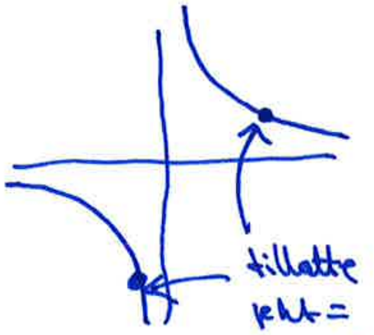
Klassifisere som lokal maks/min vha. andrederivert-testen

$(x^*, y^*) \rightsquigarrow H(f)(x^*, y^*)$

stasj. pkt

## Lagrange-problem

Eks: min $\underbrace{x^2+y^2}_{f(x,y)}$ når $\underbrace{xy=1}_{g(x,y)=a}$




alle tillatte pkt er randpkt, ingen indre pkt

Metode:

$$L = f(x,y) - \lambda \cdot g(x,y) = x^2 + y^2 - \lambda \cdot xy$$

$$\left.\begin{array}{l} L'_x = 0 \\ L'_y = 0 \\ g(x,y) = a \end{array}\right\} \text{kandidater for max/min}$$

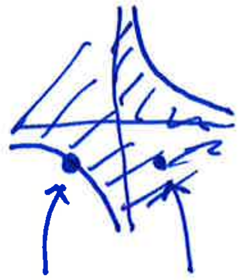
Lagrange betingelsene

Ekstremverdisetningen:

Hvis mengden D av tillatte pkt er begrenset, så fås maks/min.

## Kuhn-Tucker problem:

Eks: min $x^2+y^2$ når $xy \leq 1$



noen rand-pkt. noen indre pkt.

Metode:

(a) $xy=1$: kandidatpkt

$$\begin{array}{l} L'_x = 0 \\ L'_y = 0 \\ g(x,y) = a \end{array}$$

(b) $xy<1$: kandidatpkt

$$\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}$$

Spesielt: $xy < 1$

I tillegg: nivåkurver

Nivåkurve for f:
$f(x,y) = C$

$\boxed{\text{Stigningstallet til tangenten: } y' = -\frac{f'_x}{f'_y}}$

② Eksamensoppgaver

Eks 12/2015, Opg. 4: $f(x,y) = x^2y + xy^3 + xy^2$

a) $f'_x = 2xy + y^3 + y^2 = 0$ $\quad y(2x + y^2 + y) = 0$
$f'_y = x^2 + 3xy^2 + 2xy = 0$ $\quad x(x + 3y^2 + 2y) = 0$

$x=0,\ y=0$ : $(x,y) = (0,0)$

$y=0,\ x + 3y^2 + 2y = 0$ : $x = 0 \to (0,0)$

$2x + y^2 + y = 0,\ x = 0$ : $y^2 + y = 0$
$y(y+1) = 0$
$y = 0$ eller $y = -1$ $\quad \Big\}\ (0,0)\ (0,-1)$

$2x + y^2 + y = 0,$
$x + 3y^2 + 2y = 0$ :

$x = -3y^2 - 2y$
$\downarrow$
$2\cdot(-3y^2 - 2y) + y^2 + y = 0$
$-5y^2 - 3y = 0$
$y(-5y - 3) = 0$
$y = 0$, $x = 0$ eller $-5y - 3 = 0$
$y = -3/5$

$x = -3(-3/5)^2 - 2(-3/5)$
$= \frac{-27}{25} + \frac{6\cdot 5}{5\cdot 5}$
$= 3/25$

$(x,y) = (3/25, -3/5)$

Stasjonære pkt: $(x,y) = (0,0), (0,-1), (3/25, -3/5)$

$$2\cdot\frac{3}{25} + 3\cdot\left(-\frac{3}{5}\right)^2 + 2\cdot\left(-\frac{3}{5}\right) = \frac{6}{25} + \frac{27}{25} - \frac{6}{5}\cdot\frac{5}{5} = 3/25$$

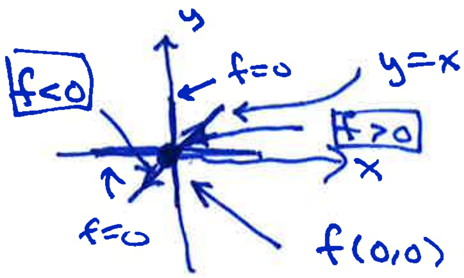
b) Er $(0,0)$ et sadelpkt?

$$\left.\begin{array}{ll} f''_{xx} = 2y & f''_{xy} = 2x + 3y^2 + 2y \\ & f''_{yy} = 6xy + 2x \end{array}\right\} \quad H(f) = \begin{pmatrix} 2y & 2x+3y^2+2y \\ 2x+3y^2+2y & 6xy+2x \end{pmatrix}$$

$$\frac{-54+30}{125} = 6\cdot\frac{3}{25}\cdot\left(-\frac{3}{5}\right) + \frac{6}{25}\cdot\frac{5}{5}$$

$$H(f)(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Ingen konklusjon fra andrederivert-testen.




$f(0,0) = 0$

$$f(x,y) = x^2y + xy^3 + xy^2$$

$y=x$: $f(x,x) = x^3 + x^4 + x^3$
$= 2x^3 + x^4$
$= x^3(2+x)$

x nær null: pos. $(x>0)$, neg. $(x<0)$; pos

$(x,y) = (0,0)$ er sadelpkt fordi det fins pkt med

$f(x,y) > f(0,0) = 0$

og

$f(x,y) < f(0,0) = 0$

i nærheten av $(0,0)$.

c) $(0,0)$: sadelpkt fra b)

$(0,-1)$: $H(f)(0,-1) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \det = AC - B^2 = -1 < 0$ sadelpkt

$(3/25, -3/5)$: $H(f)(3/25,-3/5) = \begin{pmatrix} -6/5 & 3/25 \\ 3/25 & -24/125 \end{pmatrix}$ $\quad AC - B^2 = \frac{6\cdot 24}{5\cdot 125} - \frac{9}{25\cdot 25} > 0$

lokalt maks.

Lokalt maks: $(x,y) = (3/25, -3/5)$

Ingen lokale min.

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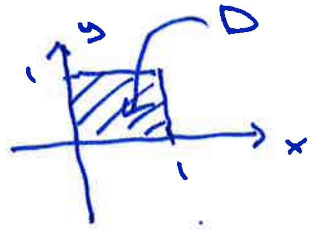
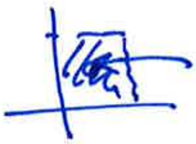



d) $D = \{(x,y) : 0 \le x,y \le 1\}$

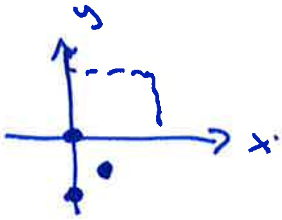
finn maks/min for $f$ på $D$.

maks/min $f(x,y)$ når $\begin{cases} x \ge 0 \\ x \le 1 \\ y \ge 0 \\ y \le 1 \end{cases}$

Kuhn-Tucker problem.

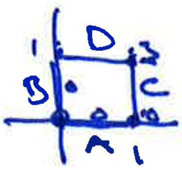
Indre pkt:  $(x,y)$

$\left.\begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array}\right\}$ stasjonære pkt.
$(0,0)$, $(0,-1)$, $(3/25, -3/5)$




ingen av disse er indre pkt

ingen kandidater her




Randpkt: $f(x,y) = x^2y + xy^3 + xy^2$

pga ekstremverdi-setningen (D er begrenset)

A: $0 \le x \le 1$, $y = 0$ : $f(x,y) = 0$

B: $x = 0$, $0 \le y \le 1$ : $f(x,y) = 0$

C: $x = 1$, $0 \le y \le 1$ : $f = y + y^3 + y^2$
$(y + y^3 + y^2)' = 1 + 3y^2 + 2y > 0$ $\Bigg\}$ min: $f = 0$ i $y = 0$; maks: $f = 3$ i $y = 1$

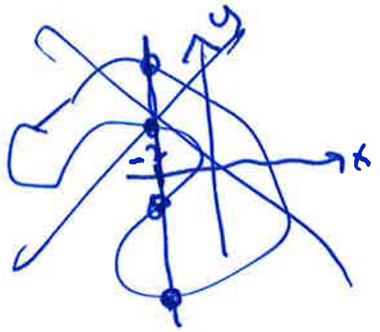
D: $y = 1$, $0 \le x \le 1$ : $f = x^2 + x + x = x^2 + 2x$
$(x^2 + 2x)' = 2x + 2 > 0$ $\Bigg\}$ min: $f = 0$ i $x = 0$; maks: $f = 3$ i $x = 1$

Maks: $f = 3$ i $(x,y) = (1,1)$

Min: $f = 0$ i $(x,y) = (0,0)$ (+ A og B)

**Opps. 5.** $g(x,y) = x^3 + xy + y^2$

a) **Nivåkurven $g(x,y)=0$:** $x^3 + xy + y^2 = 0$

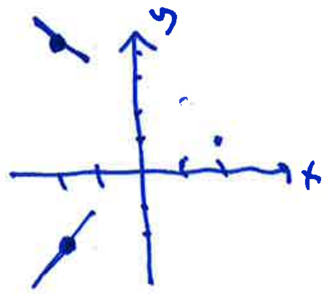



**$x=-2$:**

$$(-2)^3 + (-2)y + y^2 = 0$$

$$y^2 - 2y - 8 = 0$$

$$y = \frac{2 \pm \sqrt{4 - 4\cdot(-8)}}{2} = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2}$$




$\underline{y=4}$ eller $\underline{y=-2}$

**Konkl:** $\underline{(-2,4)}$, $\underline{(-2,-2)}$

**Tangent:** $y' = -\dfrac{g'_x}{g'_y} = -\dfrac{3x^2+y}{x+2y}$ $\begin{cases} y'(-2,4) = -\dfrac{16}{6} = -\dfrac{8}{3} \\ y'(-2,-2) = -\dfrac{10}{-6} = \dfrac{5}{3} \end{cases}$

**Tangentlinjer:**

$(-2,4)$: $y - 4 = -\frac{8}{3}(x+2) \Rightarrow y = -\frac{8}{3}x - \frac{16}{3} + \frac{12}{3} = \underline{-\frac{8}{3}x - \frac{4}{3}}$

$(-2,-2)$: $y + 2 = \frac{5}{3}(x+2) \Rightarrow y = \frac{5}{3}x + \frac{10}{3} - \frac{6}{3} = \underline{\frac{5}{3}x + \frac{4}{3}}$

b) max $f(x,y) = x$ når $\underbrace{x^3 + xy + y^2 = 0}_{g(x,y)=0}$ fra a)

Lagrange:

$\mathcal{L} = x - \lambda \cdot (x^3 + xy + y^2)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1 - \lambda \cdot (3x^2 + y) = 0 \\ \mathcal{L}'_y &= -\lambda \cdot (x + 2y) = 0 \\ & x^3 + xy + y^2 = 0 \end{aligned}}$$

Lagrange-betingelsene

2 $\longrightarrow \lambda = 0$ eller $x + 2y = 0$

$\lambda = 0$: $1 - 0 \cdot * = 0$, $1 = 0$ umulig

$x + 2y = 0$: $\boxed{x = -2y}$

1 $1 - \lambda \cdot (3 \cdot (-2y)^2 + y) = 0$

$1 - \lambda \cdot (12y^2 + y) = 0 \Rightarrow \boxed{\lambda = \frac{1}{12y^2 + y}}$

$= \frac{1}{12 \cdot 1/64 - 1/8} \cdot \frac{64}{64} = \frac{64}{12 - 8} = \frac{64}{4} = 16$

3 $(-2y)^3 + (-2y) \cdot y + y^2 = 0$

$-8y^3 - 2y^2 + y^2 = 0$

$-8y^3 - y^2 = 0$

$y^2(-8y - 1) = 0$

~~$y = 0$, $x = 0$, umulig pga $\lambda$~~ eller $-8y - 1 = 0$

$y = -1/8$

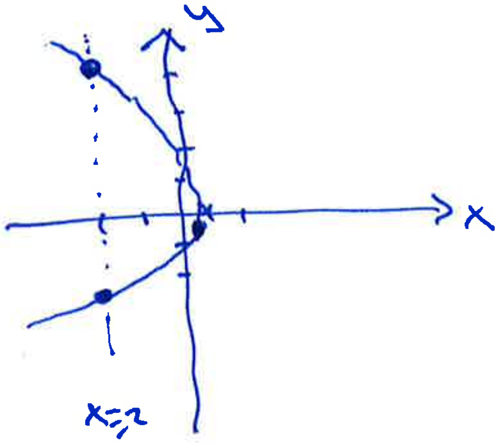
$x = 1/4$

$\boxed{\begin{aligned} x &= 1/4 \\ y &= -1/8 \\ \lambda &= 16 \end{aligned}}$ Eneste kandidat-pkt.

$\boxed{f = 1/4}$

Spørsmål om $f(1/4, -1/8) = 1/4$ er maks:

Husk: max $f(x,y) = x$ når $x^3 + xy + y^2 = 0$




Ekstremverdisetningen:

Er $g(x,y)=0$ begrenset?

Finn det pkt med $x=a$:

$x=a$ inn i: $x^3+xy+y^2=0$

$$a^3 + ay + y^2 = 0$$

$$y^2 + ay + a^3 = 0$$

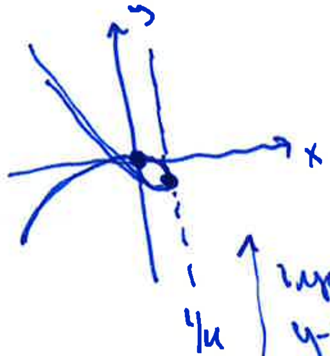
$$A=1 \quad B=a \quad C=a^3 \longrightarrow y = \frac{-a \pm \sqrt{a^2 - 4\cdot 1 \cdot a^3}}{2\cdot 1} = \frac{-a \pm \sqrt{a^2-4a^3}}{2}$$

$\boxed{a^2 - 4a^3 = 0}$: kun én y-verdi

$$a^2(1-4a) = 0$$

$a=0$, $a=1/4$

$x=0$, $y=0$ $\qquad$ $x=1/4$, $y=-1/8$




ingen y-verdier

$\boxed{a^2-4a^3<0}$: ingen y-verdier:

$$a^2(1-4a) < 0$$

| | 0 | 1/4 | |
|---|---|---|---|
| $a^2$ | | | |
| $1-4a$ | | | |
| VS | | | |

for $a > 1/4$

Konklusjon:

$f_{maks} = 1/4$ i $(1/4, -1/8)$

(men $g(x,y)=0$ er ikke begrenset)