Plan
1 Bestemte integral og antiderivasjon
2 Arealberegning og bestemte integral
3 Uegentlige integral

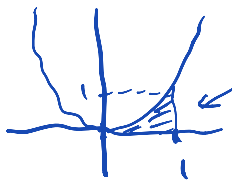
① Bestemte integral og anti-derivasjon

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{der} \quad F'(x) = f(x)$$

$=$

Eks: $\displaystyle\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3 + C\right]_0^1 = \left(\tfrac{1}{3}\cdot 1^3 + C\right) - \left(\tfrac{1}{3}\cdot 0^3 + C\right)$ (x=1, x=0)

$$= \tfrac{1}{3} + \cancel{C} - 0 - \cancel{C} = \underline{\underline{1/3}}$$



Eks: $\displaystyle\int_1^2 1\cdot \ln x\,dx = \left[x \ln x - x\right]_1^2 = (2\ln 2 - 2) - (1\cdot \cancel{\ln 1} - 1) = \underline{\underline{2\ln 2 - 1}}$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= \left[x \cdot \ln x - \int x \cdot \tfrac{1}{x}\,dx\right]_1^2$$

$$= \left[x \ln x - x + C\right]_1^2 = (2\ln 2 - 2) - (-1) = \underline{2\ln 2 - 1}$$

Eks:

$$\int_0^1 x\cdot\sqrt{x^2+1}\,dx = \left[\int x\cdot\sqrt{u}\,\frac{du}{2\cancel{x}}\right]_{x=0}^{x=1}$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

$x=0:\ u=1$
$x=1:\ u=2$

$$= \left[\frac{1}{2}\int u^{1/2}\,du\right]_{x=0}^{x=1} = \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_{x=0}^{x=1}$$

$$= \left[\frac{1}{3}(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}2^{3/2} - \frac{1}{3}\cdot 1^{3/2}$$

$$= \frac{1}{3}\cdot 2\sqrt{2} - \frac{1}{3}\cdot 1 = \underline{\underline{\frac{1}{3}(2\sqrt{2}-1)}}$$

Alt:

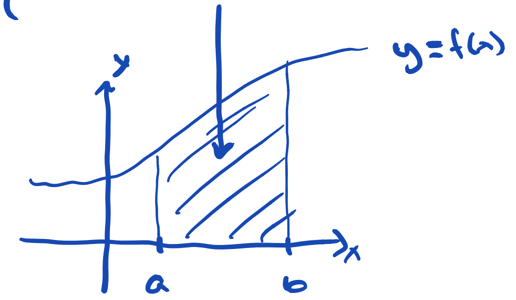
$$\left[\int x\sqrt{u}\cdot\frac{du}{2\cancel{x}}\right]_{u=1}^{u=2} = \int_1^2 \frac{1}{2}u^{1/2}\,du$$

$$= \left[\frac{1}{3}u^{3/2}\right]_1^2 = \frac{1}{3}\cdot 2^{3/2} - \frac{1}{3}\cdot 1^{3/2} = \frac{1}{3}\underline{\underline{(2\sqrt{2}-1)}}$$

Teorem:

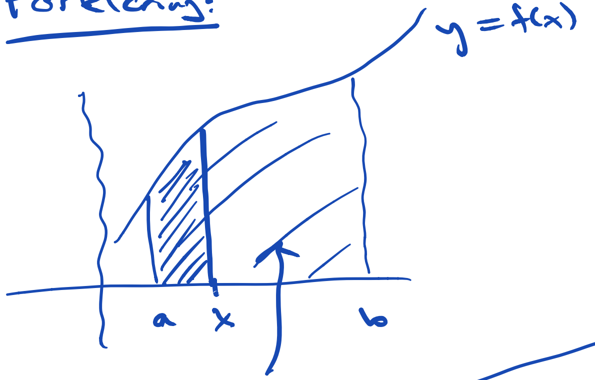
Hvis $f(x)$ er en kontinuerlig funksjon på $[a,b]$ og $f(x)\geq 0$ for $a\leq x\leq b$, så er

$$\left\{\begin{array}{l}\text{arealet under grafen}\\ \text{til } f(x) \text{ for } a\leq x\leq b\end{array}\right\} = \left\{\begin{array}{l}F(b)-F(a)\\ \text{når } F'(x)=f(x)\end{array}\right\} = \int_a^b f(x)\,dx$$



Vi kan regne ut arealet under grafen til en kontinuerlig funksjon ved å bruke antiderivert.

Forklaring:

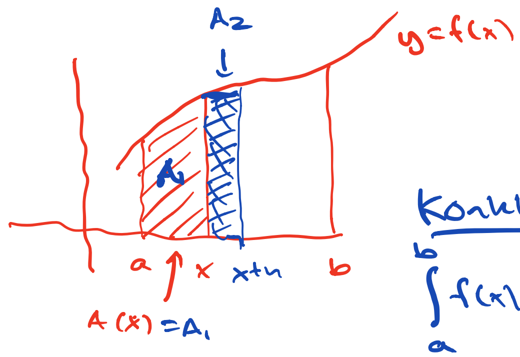


A = areal

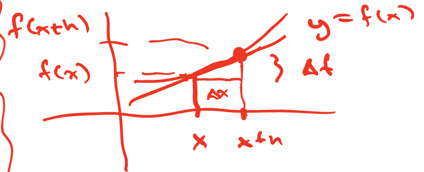
$A(x)$ = areal under grafen til $f(x)$ mellom a og x.

$A(b) = \underline{A}$ $\qquad A(a) = \underline{0}$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \approx \frac{(A_1 + A_2) - A_1}{h} = \frac{A_2}{h} \approx \frac{\cancel{h} \cdot f(x)}{\cancel{h}} = f(x)$$



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{\Delta f}{\Delta x}$$

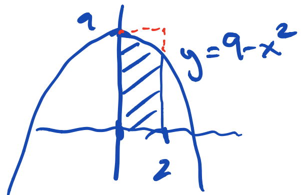


Konklusjon: $\boxed{A'(x) = f(x)}$

$$\int_a^b f(x)\,dx = \Big[A(x)\Big]_a^b = A(b) - A(a) = A - 0 = \underline{\underline{A}}$$

② Arealberegning og bestemte integral

Eks: Arealet under $y = 9 - x^2$ for $0 \le x \le 2$:



$$A = \int_0^2 9 - x^2\,dx = \Big[9x - \tfrac{1}{3}x^3\Big]_0^2$$

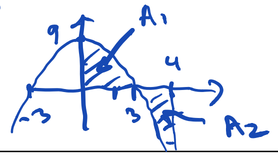
$$= \left(9 \cdot 2 - \tfrac{1}{3} \cdot 2^3\right) - (0)$$

$$= 18 - \frac{8}{3} = \frac{46}{3} \approx \underline{\underline{15{,}33}}$$

$$\boxed{\text{arealet under grafen til } f(x) = \int_a^b f(x)\,dx}$$

$$\int_0^4 9 - x^2\,dx = \Big[9x - \tfrac{1}{3}x^3\Big]_0^4 = \left(9 \cdot 4 - \tfrac{1}{3} \cdot 4^3\right) - 0 = 36 - \frac{64}{3} \approx \underline{\underline{14.67}}$$

$$= \int_0^3 9 - x^2\,dx + \int_3^4 9 - x^2\,dx = A_1 - A_2$$



$$= A_1 \qquad - A_2$$

Arealberegning:

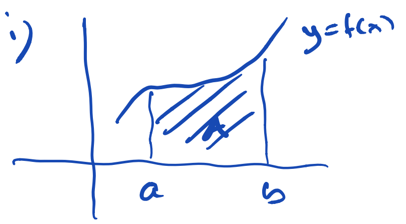
Anta $f(x)$ kontinuerlig på $[a,b]$

i) $f(x) \geq 0$ for $a \leq x \leq b$: $\int_a^b f(x)\,dx = A \leftarrow$ {A = arealet under grafen i $[a,b]$}

ii) $f(x) \leq 0$ — " — : $\int_a^b f(x)\,dx = -A \leftarrow$ {A = arealet mellom grafen til f og x-aksen}

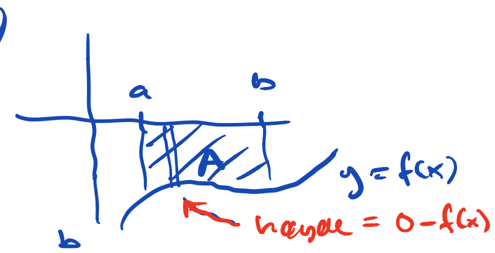
iii) $f(x) \geq g(x)$ — " — : $\int_a^b f(x)-g(x)\,dx = A \leftarrow$ {A = arealet mellom $y=f(x)$ og $y=g(x)$}

i)



$$\int_a^b f(x)\,dx = A$$

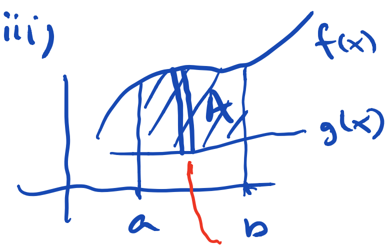
ii)



$$\int_a^b f(x)\,dx = -A$$

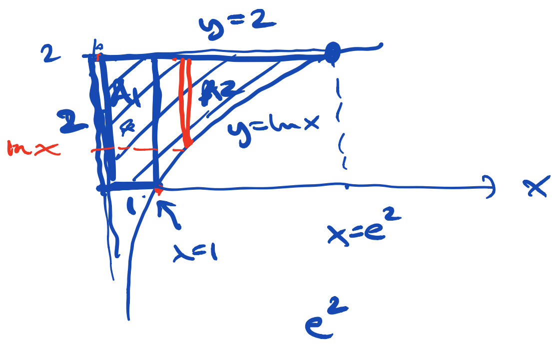
$$\int_a^b -f(x)\,dx = A$$

iii)



$$\int_a^b f(x)-g(x)\,dx = A$$

Eks: Finn arealet begrenset av $y = \ln x$, $y=2$ og y-aksen, og x-aksen



Skjæring: $y=0$ og $y = \ln x$

$0 = \ln x \quad | e^{\cdot}$

$e^0 = x \qquad \underline{x=1}$

$y=2$ og $y = \ln x$

$2 = \ln x \quad | e^{\cdot}$

$e^2 = x \qquad \underline{x = e^2}$

$$A = \underbrace{1\cdot 2}_{A_1} + \underbrace{\int_1^{e^2} (2-\ln x)\,dx}_{A_2}$$

$$= 2 + \left[2x - (x\ln x - x)\right]_1^{e^2}$$

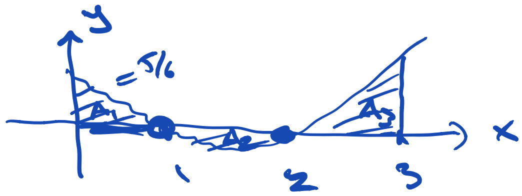
$$= 2 + \left[3x - x\ln x\right]_1^{e^2} = 2 + (3e^2 - e^2 \ln e^2) - (3 - 1\cdot \cancel{\ln 1})$$

$$= 2 + 3e^2 - e^2\cdot 2 - 3 = \underline{e^2 - 1}$$

Eks: $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$

Finn arealet mellom grafen $y = f(x)$ og x-aksen i intervallet $[0,3]$.

$$\boxed{\int_0^3 f(x)\,dx = A_1 - A_2 + A_3 \neq A}$$



$$A = \underset{f(x)\geq 0}{A_1} + \underset{f(x)\leq 0}{A_2} + \underset{f(x)\geq 0}{A_3} = \int_0^1 x^2 - 3x + 2\,dx - \int_1^2 x^2 - 3x + 2\,dx + \int_2^3 x^2 - 3x + 2\,dx$$

$$= \left[\frac{1}{3}x^3 - 3\cdot\frac{1}{2}x^2 + 2x\right]_0^1 - \left[\cdots\right]_1^2 + \left[\cdots\right]_2^3$$

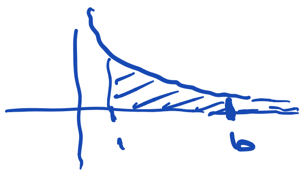
$$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - 0 \quad - \cdots \quad + \cdots$$

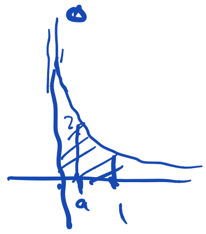
$$= \frac{2}{6} - \frac{9}{6} + \frac{12}{6} \quad - \cdots + \cdots$$

$$= \frac{5}{6} - \cdots + \cdots =$$

3 Uegentlige integraler

Ex: $\int_1^\infty \frac{1}{x^2}\,dx$ $\qquad \int_1^\infty \frac{1}{x}\,dx$ $\qquad \int_0^1 \frac{1}{x}\,dx$

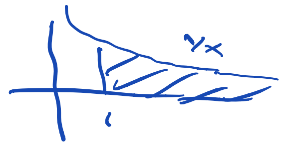




Ex:

$$\int_1^\infty \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^\infty = \lim_{b\to\infty}\Big[\ln|x|\Big]_1^b$$

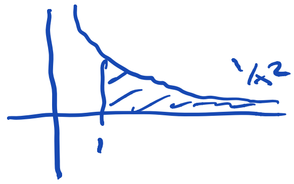
$$= \lim_{b\to\infty}(\ln b) = \underline{\underline{\infty}}$$



$$\int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^\infty$$

$$= \lim_{b\to\infty}\underline{\underline{\left[-\frac{1}{x}\right]_1^b}} = \lim_{b\to\infty}\left(-\frac{1}{b}-\left(-\frac{1}{1}\right)\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = \underline{\underline{1}}$$

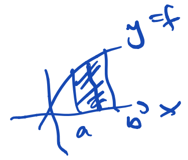


Del 2: – Kort repetisjon
– oppgavegjennomgang Oppgaveark 19

Kort repetisjon: $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$

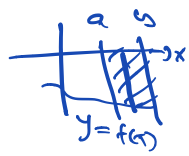
når $F'(x) = f(x)$.

Arealberegning: i) $f(x) \geq 0$ i $[a,b]$: $A = \int_a^b f(x)\,dx$

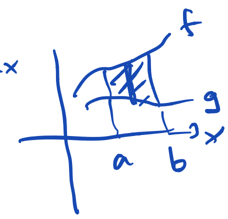


ii) $f(x) \leq 0$ –"– $A = -\int_a^b f(x)\,dx$

$= \int_a^b -f(x)\,dx$



iii) $f(x) \geq g(x)$ i $[a,b]$: $A = \int_a^b f(x) - g(x)\,dx$



Oppgaver, Oppgaveark 19:

2f: $\int_{-1}^{1} \underbrace{x \ln(x^2+1)}_{f(x)}\,dx = \int_2^2 \cancel{x} \cdot \ln(u) \frac{du}{2\cancel{x}} = \int_2^2 \frac{1}{2} \ln(u)\,du$

$\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$ $x=-1: u=2$, $x=1: u=2$

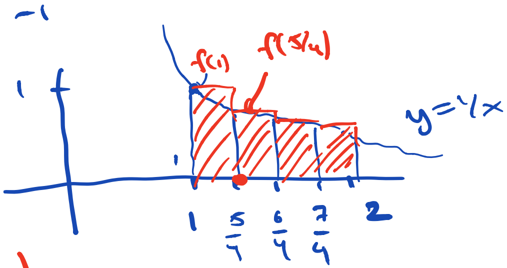
$= \left[\frac{1}{2}\left(u \ln(u) - u\right)\right]_2^2 = \underline{\underline{0}}$

$\int_{-1}^{1} \underbrace{x \ln(x^2+1)}_{f(x)}\,dx = \underline{\underline{0}}$

Ser at $f(-x) = -f(x)$

$\int_{-1}^{1} f(x)\,dx = -A_1 + A_2 = 0$

3b. $\int_1^2 \frac{1}{x}\,dx$ tilnærmet vha Riemann-sum, $\underline{n=4}$



$R.S. = \frac{1}{4} \cdot f(1) + \frac{1}{4} \cdot f(5/4) + \frac{1}{4} \cdot f(6/4) + \frac{1}{4} \cdot f(7/4)$

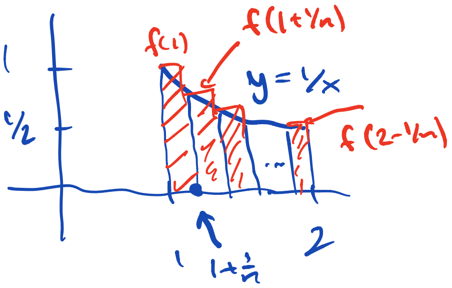
$\Delta x = \frac{2-1}{4} = 1/4 = \underline{0.25}$

$= \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{4} + \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{5} + \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{6} + \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{7}$

$= \underline{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}$

$\int_1^2 \frac{1}{x}\,dx = \left[\ln x\right]_1^2 = \ln 2 - \cancel{\ln 1} = \underline{\underline{\ln 2}}$

8. $\int_1^2 \frac{1}{x}\,dx$ Vis at $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\dots+\frac{1}{2n-1}\right)=\ln 2$



① $A = \ln 2$ vha antiderivasjon

② A vha Riemannsum, n delepkt.

$\Delta x = \frac{2-1}{n} = 1/n$

RS: $\frac{1}{n}\cdot f(1) + \frac{1}{n}\cdot f(1+1/n) + \dots + \frac{1}{n}\cdot f(2-1/n)$

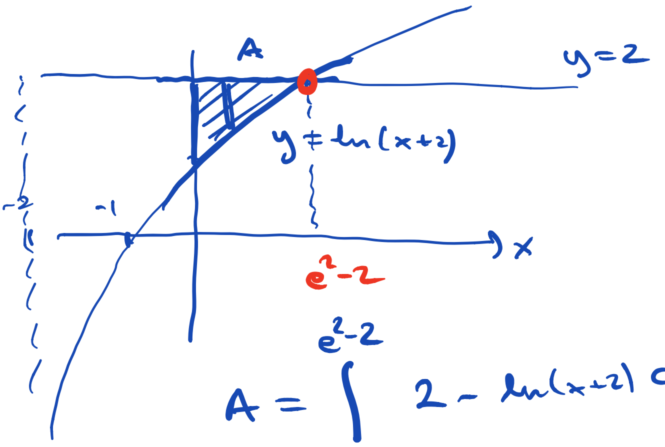
$= \frac{1}{n}\cdot\frac{n}{n} + \frac{1}{n}\cdot\frac{1}{1+1/n} + \dots + \frac{1}{n}\cdot\frac{1}{2-1/n}$

$= \left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1}\right)$

$\downarrow$ når $n \to \infty$

$\ln 2$

5. Areal begrenset av: $y = \ln(x+2)$, $y = 2$, y-aksen



Skjæring: $y = \ln(x+2)$, $y = 2$

$2 = \ln(x+2) \quad | e$

$e^2 = x+2$

$x = e^2 - 2$

$$A = \int_0^{e^2-2} 2 - \ln(x+2)\,dx$$

$$= \left[2x - \left((x+2)\ln(x+2) - (x+2)\right)\right]_0^{e^2-2}$$

$$= \left(2(e^2-2) - e^2 \underbrace{\ln e^2}_{2} + e^2\right) - \left(0 - 2\ln 2 + 2\right)$$

$$= \cancel{2e^2} - 4 - \cancel{e^2\,2} + e^2 + 2\ln 2 - 2$$

$$= e^2 + 2\ln 2 - 6$$

$\int \ln x\,dx = x\ln x - x + C$ ✓

$1 \cdot \ln x$

delvis

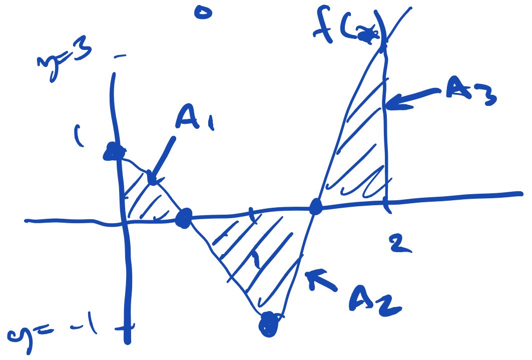
$\int \ln(x+2)\,dx = \int \ln(u)\,du$

$\boxed{u = x+2,\; du = 1\cdot dx}$

$= u\ln u - u + C$

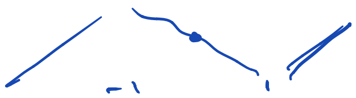
$= (x+2)\ln(x+2) - (x+2) + C$

4. $\displaystyle\int_0^2 \underbrace{x^3 - 3x + 1}\, dx = A_1 - \underline{A_2} + A_3$



$f(0) = 1 \quad f(2) = 3 \quad f(1) = -1$

$f'(x) = 3x^2 - 3 = 3(x^2-1)$
$= 3(x-1)(x+1)$



7e. $\displaystyle\int_0^{\infty} \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\, dx = \int_1^{-\infty} \frac{e^u}{\cancel{\sqrt{x}}}(-2\cancel{\sqrt{x}})\, du = \int_1^{-\infty} -2e^u\, du$

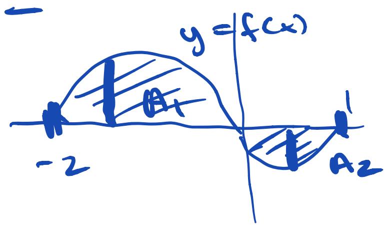
$\boxed{\begin{aligned} u &= 1-\sqrt{x} \\ du &= -\tfrac{1}{2\sqrt{x}}\, dx \end{aligned}}$

$x = 0: \ u = 1$
$x = \infty: \ u = -\infty$

$= \left[-2e^u\right]_1^{-\infty} = \lim_{b\to-\infty} \left[-2e^u\right]_1^b = \lim_{b\to-\infty} \left(-2e^b + 2e^1\right)$

$= 2e - 0 = \underline{\underline{2e}}$ $\qquad b \to -\infty \Rightarrow e^b \to 0$

9d.



$A_2 = 22/15$

$\displaystyle\int_{-2}^{1} f(x)\,dx = \frac{18}{5}$

$\parallel$

$A_1 - A_2 = 18/5$

$A_1 = 18/5 + A_2 = 18/5 + 22/15$

$= \dfrac{3\cdot 18 + 22}{15} = \underline{\underline{76/15}}$

9c. $\displaystyle\int \frac{6-3x}{4-9x^2}\,dx = \int \frac{1}{2-3x} + \frac{2}{2+3x}\,dx$

$= \int \frac{1}{u}\cdot\frac{du}{-3} + \int \frac{2}{v}\cdot\frac{dv}{3}$

$\boxed{\begin{array}{l} u = 2-3x \\ du = -3\,dx \end{array}}$ $\boxed{\begin{array}{l} v = 2+3x \\ dv = 3\,dx \end{array}}$

$= -\frac{1}{3}\ln|2-3x| + \frac{2}{3}\ln|2+3x| + C$

$= \frac{1}{3}\left(2\ln|2+3x| - \ln|2-3x|\right) + C$

$= \frac{1}{3}\ln\left|\frac{(2+3x)^2}{2-3x}\right| + C$

Delbrøksoppspalting:

$4-9x^2 = (2-3x)(2+3x)$

$\frac{6-3x}{4-9x^2} = \boxed{\frac{A}{2-3x} + \frac{B}{2+3x}}$ $\Big| \cdot (2-3x)(2+3x)$

$6-3x = A\cdot(2+3x) + B(2-3x)$

$6-3x = (2A+2B) + (3A-3B)x$

$2A+2B = 6$, $3A-3B = -3$

$2A + 2B = 6$
$3A - 3B = -3$

$A + B = 3$
$A - B = -1$
$2A = 2$

$A = 1$
$B = 2$

$4-9x^2 = \,?$

$= -9\left(x-\frac{2}{3}\right)\left(x+\frac{2}{3}\right)$

$4-9x^2 = 0$
$-9x^2+4 = 0$
$x^2 = 4/9$
$x = \pm 2/3$