Plan

1 Substitusjon
2 Delvis integrasjon
3 Delbrøksoppspaltning

Forrige uke:
- regneregler for integrasjon
- substitusjon

1 Substitusjon:

$$\int f(x)\,dx = \int g(u)\,du$$

$$\boxed{\begin{aligned} u &= u(x) \\ du &= u'(x)\,dx \end{aligned}}$$

$$\Downarrow \quad dx = \frac{1}{u'(x)}\,du$$

Ex:

$$\int x e^{-x^2}\,dx = \int \cancel{x} e^u \frac{1}{(-2\cancel{x})}\,du = -\frac{1}{2}\int e^u\,du$$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \end{aligned}} \quad \Downarrow \quad dx = \frac{1}{-2x}\,du$$

$$= -\frac{1}{2}\left(e^u\right) + C = \underline{\underline{-\frac{1}{2}e^{-x^2} + C}}$$

Ex:

$$\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}}\,\underbrace{\cancel{x}\,du} = \int u\,du$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}} \quad \Downarrow \quad dx = x\,du$$

Potensregelen $n=1$

$$= \frac{1}{2}u^2 + C = \underline{\underline{\frac{1}{2}(\ln x)^2 + C}}$$

Ex:

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^{1-u}}{u}\,\underbrace{2\sqrt{x}\,du} = \int \frac{e^{1-u}}{\cancel{u}}\,2\cancel{u}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}} \quad \Downarrow \quad dx = 2\sqrt{x}\,du$$

$$= 2\int e^{1-u}\,du$$

$$= 2\left(\frac{1}{-1}e^{1-u}\right) + C$$

$$= -2e^{1-u} + C = \underline{\underline{-2e^{1-\sqrt{x}} + C}}$$

$$\boxed{\begin{aligned} v &= 1-u \\ dv &= -1\cdot du \end{aligned}}$$

$$= 2\int e^v \frac{1}{(-1)}\,dv$$

$$= -2e^v + C$$

$$= -2e^{1-u} + C$$

Ex: $\int e^{\sqrt{x}}\,dx = \int e^u \, 2\sqrt{x}\,du = \int e^u\, 2u\,du$

$$\boxed{\begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$$

$\Downarrow$

$dx = 2\sqrt{x}\,du$

$= \int 2u \cdot e^u\,du$

delvis int. (se nedenfor)

$= 2ue^u - 2e^u + C$

$= \underline{\underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}}$

② Delvis integrasjon: "Produktregel for integrasjon"

$$\boxed{\int u' \cdot v\,dx = u \cdot v - \int u \cdot v'\,dx}$$

Forklaring:

Derivasjon:

$(uv)' = u'v + uv'$

$\Downarrow$

$\int (uv)'\,dx = \int u'v + uv'\,dx$

$uv = \int u'v\,dx + \int uv'\,dx$

$\int u'v\,dx = uv - \int uv'\,dx$

Ex: $\int \underset{u'}{x} \cdot \underset{v}{\ln x}\,dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C = \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$

Ex: $\int 2x \cdot e^x\,dx = e^x \cdot 2x - \int e^x \cdot 2\,dx$

$$\cancel{\boxed{\begin{array}{ll} u = x^2 & v = e^x \\ u' = 2x & v' = e^x \end{array}}}$$

$$\boxed{\begin{array}{ll} u = e^x & v = 2x \\ u' = e^x & v' = 2 \end{array}}$$

$= 2xe^x - 2\int e^x\,dx = \underline{\underline{2xe^x - 2e^x + C}}$

Ex: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx =$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 \, dx$$
$$= x \ln x - x + C$$

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

③ Integrasjon av rasjonale funksjoner (brøkuttrykk)

Eks: i) $\int \frac{1}{1-x} dx$ ii) $\int \frac{x}{1-x} dx$ iii) $\int \frac{2x}{1-x^2} dx$ iv) $\int \frac{2}{1-x^2} dx$

i) $\int \frac{1}{1-x} dx = \int \frac{1}{u} \cdot \frac{1}{(-1)} du = -\ln|u| + C$

$$\boxed{\begin{array}{l} u = 1-x \\ du = -1 \cdot dx \end{array}}$$

$$= -\ln|1-x| + C$$

$$\int \frac{A}{ax+b} dx = \int \frac{A}{u} \cdot \frac{1}{a} du = \frac{A}{a} \cdot \int \frac{1}{u} du$$

$$\boxed{\begin{array}{l} u = ax+b \\ du = a \cdot dx \end{array}}$$

$$= \frac{A}{a} \ln|u| + C = \frac{A}{a} \ln|ax+b| + C$$

$$\boxed{\int \frac{A}{ax+b} dx = \frac{A}{a} \ln|ax+b| + C \quad (a \neq 0)}$$

ii) $\int \frac{x}{1-x} dx = \int -1 + \frac{1}{1-x} dx = -x + \frac{1}{-1} \cdot \ln|1-x| + C$

$$= -x - \ln|1-x| + C$$

polyn.div:

$x : (-x+1) = -1$
$-(x-1)$
1

$\frac{x}{1-x} = -1 + \frac{1}{1-x}$

iii) $\displaystyle\int \frac{2x}{1-x^2}\,dx = \int \frac{\cancel{2x}}{u}\frac{1}{(-\cancel{2x})}\,du = -\int \frac{1}{u}\,du$

$\boxed{\begin{array}{c} u = 1-x^2 \\ du = -2x\,dx \end{array}}$

$= -\ln|u| + C = \underline{\underline{-\ln|1-x^2| + C}}$

iv) $\displaystyle\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx = \underline{\underline{\ln|1+x| - \ln|1-x| + C}}$

<u>Delbrøksoppspaltning:</u>

$\dfrac{2}{1-x^2} = \dfrac{A^{=1}}{1+x} + \dfrac{B^{=1}}{1-x}$ $\quad | \cdot (1-x^2)$ $\quad A, B$ konstanter (ukjente)

<u>Metode:</u>

i) $1-x^2 = (1-x)(1+x)$

$\Rightarrow \dfrac{2}{1-x^2} = \dfrac{A}{1+x} + \dfrac{B}{1-x}$

ii) Finn A og B

$\dfrac{2}{\cancel{1-x^2}} \cdot \cancel{(1-x^2)} = \dfrac{A}{\cancel{1+x}} \cdot (1-x)\cancel{(1+x)} + \dfrac{B}{\cancel{1-x}} \cdot \cancel{(1-x)}(1+x)$

$2 = A(1-x) + B(1+x)$

Alt I: sammenlikne koeff.

$2 = A - Ax + B + Bx$

$\underline{0} \cdot x + \underline{\underline{2}} = \underline{\underline{(A+B)}} + \underline{(-A+B)}x$

$0 = -A + B \rightarrow A = B$

$2 = A + B \qquad 2A = 2$

$\underline{A=1} \quad \underline{B=1}$

Alt. 2

$\boxed{2 = A(1-x) + B(1+x)}$

$\underline{x=1}$: $2 = A \cdot 0 + B \cdot 2 \Rightarrow \underline{B=1}$

$\underline{x=-1}$: $2 = A \cdot 2 + B \cdot 0 \qquad \underline{A=1}$