Plan

1 Delbrøkoppspaltning
2 Bestemte integral

Integrasjon av rasjonale uttrykk: $\frac{p(x)}{q(x)}$ deg $p(x)$ deg $q(x)$

Enkleste tilfelle: deg $q(x) = 1$

Eks: $\int \frac{x^2}{1-x}$

– polynomdivisjon
hvis deg $p(x) \geq$ deg $q(x)$
(teller ikke konstant)

① $x^2 : -x+1 = \underline{-x-1}$

$$\begin{array}{l} x^2 - x \\ \hline x \\ x-1 \\ \hline 1 \end{array}$$

$$\frac{x^2}{1-x} = -x-1+\frac{1}{1-x}$$

– substitusjon.
u = nevner

$$\boxed{\begin{aligned} u &= q(x) \\ du &= q'(x) \cdot dx \end{aligned}}$$

$$\int \frac{x^2}{1-x}\,dx = \int -x-1+\frac{1}{1-x}\,dx$$
$$= -\frac{1}{2}x^2 - x + \int \frac{1}{1-x}\,dx$$
$$= -\frac{1}{2}x^2 - x + \frac{1}{-1}\ln|1-x| + C$$
$$= -\frac{1}{2}x^2 - x - \ln|1-x| + C$$

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C \quad (a \neq 0)$$

Vanskeligere tilfelle : deg $q(x) \geq 2$

(graden til nevner)

Eks: $\int \frac{1}{x^2}\,dx \neq \ln(x^2) + C$ ← kan ikke bruke substitusjon her

$\int \frac{2}{(1+x)^2}\,dx \neq 2\ln(1+x)^2 + C$

Eks: $\int \frac{2}{(1+x)^2}\,dx = \int \frac{2}{u^2}\,du = \int 2u^{-2}\,du$

$\boxed{\begin{array}{l} u = 1+x \\ du = 1\cdot dx \end{array}}$

$= 2\cdot\frac{u^{-1}}{-1} + C = -\frac{2}{u} + C$

$= -\frac{2}{1+x} + C$

$\int \frac{x-2}{x^2-4x+3}\,dx = \int \frac{x-2}{u}\,\frac{1}{2x-4}\,du$

$\boxed{\begin{array}{l} u = x^2-4x+3 \\ du = (2x-4)\,dx \end{array}} \to dx = \frac{1}{2x-4}\,du$

$= \int \frac{1}{u}\,\frac{x-2}{2x-4}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$\frac{x-2}{2x-4} = \frac{\cancel{x-2}}{2\cancel{(x-2)}}$

$= \frac{1}{2}\ln|x^2-4x+3| + C$

① Delbrøksoppspalting

$$\frac{1}{x} + \frac{2}{x+1} = \frac{1}{x}\cdot\frac{x+1}{x+1} + \frac{2}{x+1}\cdot\frac{x}{x} = \frac{x+1+2x}{x\cdot(x+1)} = \frac{3x+1}{x(x+1)}$$

Eks: $\displaystyle\int \frac{x}{x^2-4x+3}\,dx$

Delbrøksoppspalting: $\displaystyle\frac{x}{x^2-4x+3}$

① Sjekk at graden til teller < graden i nevner
$\Rightarrow$ Hvis ikke: Gjør polynomdivisjon først.

② Faktoriser nevner:

$$x^2-4x+3 = (x-1)(x-3)$$

$x^2-4x+3=0$

$$x = \frac{4\pm\sqrt{4^2-4\cdot 3}}{2} = \frac{4\pm 2}{2} = 3, 1$$

③ Skriv opp uttrykket:

$$\frac{x}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3}$$

(A, B konstanter som vi må finne)

④ Finn A og B:

$$\frac{x}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3} \quad \Big|\cdot(x-1)(x-3)$$

$$x = \frac{A}{\cancel{x-1}}\cdot\cancel{(x-1)}(x-3) + \frac{B}{\cancel{x-3}}\cdot(x-1)\cancel{(x-3)}$$

$$x = A\cdot(x-3) + B(x-1)$$

Alt I: Algebra

$$x = Ax - 3A + Bx - B$$

$$x = (A+B)x + (-3A-B)$$

$$\Downarrow$$

$$A+B = 1 \qquad A + (-3A) = 1$$

$$-3A - B = 0 \Rightarrow B = -3A$$

$$-2A = 1 \quad A = -1/2$$

$$B = 3/2$$

Alt 2: Geometri
Rett linje bestemt av 2 pkt.

$$x = A(x-3) + B(x-1)$$

$$x = 3: \; 3 = B \cdot 2 \quad B = 3/2$$

$$x = 1: \; 1 = A \cdot (-2) \quad A = -1/2$$

$$\frac{x}{x^2 - 4x + 3} = \frac{-1/2}{x-1} + \frac{3/2}{x-3}$$

$$\int \frac{x}{x^2-4x+3}\, dx = \int \frac{-1/2}{x-1} + \frac{3/2}{x-3}\, dx$$

$$= \frac{-1/2}{1} \ln|x-1| + \frac{3/2}{1} \ln|x-3| + C$$

$$= -\frac{1}{2} \ln|x-1| + \frac{3}{2} \ln|x-3| + C$$

Eks: $\displaystyle\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1-x} + \frac{1}{1+x}\,dx$

$1-x^2 = (1-x)(1+x)$

$1-x^2=0 \quad x^2=1 \quad x=\pm 1$

$-1\cdot(x-1)(x+1) = 1-x^2$

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \quad \cdot(1-x)(1+x)$$

$$2 = A(1+x) + B(1-x)$$

$x=-1$: $2 = B\cdot 2$ $\quad B=1$

$x=1$: $2 = A\cdot 2$ $\quad A=1$

$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$\ln a + \ln b = \ln(a\cdot b)$

$$= \frac{1}{-1}\ln|1-x| + \frac{1}{1}\ln|1+x| + C = \underline{\underline{\ln|1+x| - \ln|1-x| + C}}$$

$$= \underline{\underline{\ln\frac{|1+x|}{|1-x|} + C}}$$

Eks: $\displaystyle\int \frac{2x}{1-2x+x^2}\,dx = \int \frac{-2}{1-x} + \frac{2}{(1-x)^2}\,dx$

$1-2x+x^2 = (1-x)^2 = (1-x)\cdot(1-x)$

$$\frac{2x}{(1-x)^2} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} \quad \Big|\cdot(1-x)^2$$

$$2x = A(1-x) + B$$

$x=1$: $2 = B$ $\quad B=2$

$x=0$: $0 = A\cdot 1 + B$

$A = -B = \underline{-2}$

$$= \frac{-2}{-1}\ln|1-x| + \int \frac{2}{(1-x)^2}\,dx$$

$$= \underline{\underline{2\ln|1-x| + \frac{2}{1-x} + C}}$$

$$\int \frac{2}{(1-x)^2}\,dx = \int \frac{2}{u^2}\cdot\frac{1}{(-1)}\,du = -2\int u^{-2}\,du = -2\cdot\frac{u^{-1}}{(-1)} + C$$

$u = 1-x$, $du = -1\cdot dx$

$$= \frac{2}{u} + C = \frac{2}{1-x} + C$$

Eks: $\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$

Eks: $\int \frac{3}{x^3-x}\,dx =$

$$x^3 - x = x(x^2-1) = x(x+1)(x-1)$$

$$\frac{3}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$\vdots$

tilsvarende metode som når nevneren har grad 2

② Bestemte integral

Eks: $\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3 + \cancel{C}\right]_0^1 = \left(\frac{1}{3}\cdot 1^3 + C\right) - \left(\frac{1}{3}\cdot 0^3 + C\right)$

$$= \left(\frac{1}{3} + \cancel{C}\right) - \left(0 + \cancel{C}\right) = \underline{\underline{1/3}}$$

Defn: Hvis $f(x)$ er en kontinuerlig funksjon på intervallet $[a,b]$ og $F(x)$ er en antiderivert til $f(x)$, så definerer vi

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

$[F(x)]_a^b$

Eks: $\int_1^2 x^2 \ln x \, dx = \left[\frac{1}{3}x^3 \cdot \ln x\right]_1^2 - \int_1^2 \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$

Delvis:

$u = \frac{1}{3}x^3 \quad v = \ln x$

$u' = x^2 \quad v' = 1/x$

$$= \left[\frac{1}{3}x^3 \cdot \ln x\right]_1^2 - \int_1^2 \frac{1}{3}x^2 \, dx = \Big[\overbrace{\frac{1}{3}x^3 \cdot \ln x - \frac{1}{3} \cdot \frac{1}{3}x^3}^{F(x)}\Big]_1^2$$

$$= \left(\frac{1}{3} \cdot 2^3 \cdot \ln 2 - \frac{1}{9} \cdot 2^3\right) - \left(\cancel{\frac{1}{3} \cdot 1^3 \cdot \ln 1} - \frac{1}{9} \cdot 1^3\right)$$

$$= \frac{8}{3} \cdot \ln 2 - \frac{8}{9} + \frac{1}{9} = \underline{\underline{\frac{8}{3} \ln 2 - \frac{7}{9}}}$$