Plan

1 Inverse matriser
2 Indreprodukt av vektorer

Tenk:

$2 \cdot \frac{1}{2} = 1$, $\frac{1}{2} = 2^{-1}$

Husk:

$I = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & 1 \end{pmatrix}$

$A \cdot I = A$

$I \cdot A = A$

(1) Inverse matriser

A matrise

Defn. En invers matrise for A er en matrise A^{-1} slik at $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Eks: $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ A^{-1} er en matrise slik at

$A^{-1} = ? \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \cdot A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^{-1} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Oppsemark 35
Oppgave 9 a)
$AX = I$

Merk: i) Hvis A^{-1} eksisterer, (A invertibel)
så er den entydig.

Anta B, C har egenskapen at

$AB = BA = I$

$AC = CA = I$

Da har vi:

$BAC = (BA) \cdot C = IC = C$

$BAC = B \cdot (AC) = BI = B$ $\Rightarrow$ $\underline{B = C}$

ii) Hvis A^{-1} eksisterer, så har vi

$A \cdot A^{-1} = I \Rightarrow |A \cdot A^{-1}| = |I|$

$|A| \cdot |A^{-1}| = 1$

$\Rightarrow |A^{-1}| = 1/|A|$

Spesielt må $\underline{|A| \neq 0}$.

Resultat:

A er invertibel (A^{-1} fins)

$\Updownarrow$

A er kvadratisk med $|A| \neq 0$

I så fall så er A^{-1} entydig.

Metode for å finne A^{-1} når A er en $n\times n$-matrise med $|A|\neq 0$

$n=2$: $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$

$|A| = ad-bc$

$A^{-1} = \frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$ hvis $ad-bc\neq 0$

A^{-1} fins ikke hvis $ad-bc=0$

Eks: $A=\begin{pmatrix} 1 & 3 \\ 2 & 5\end{pmatrix}$

$|A| = 1\cdot 5-2\cdot 3 = -1 \neq 0$

$A^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -3 \\ -2 & 1\end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1\end{pmatrix}$

$A\cdot A^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5\end{pmatrix}\cdot\begin{pmatrix} -5 & 3 \\ 2 & -1\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$

$A^{-1}\cdot A = \begin{pmatrix} -5 & 3 \\ 2 & -1\end{pmatrix}\cdot\begin{pmatrix} 1 & 3 \\ 2 & 5\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$

Eks. $\begin{aligned} x+3y &= 14 \\ 2x+5y &= -17\end{aligned}$

$\begin{pmatrix} 1 & 3 \\ 2 & 5\end{pmatrix}\cdot\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} 14 \\ -17\end{pmatrix}$ matriseform

$A\cdot \underline{x} = \underline{b} \quad | A^{-1}\cdot$

$A^{-1}\cdot(A\underline{x}) = A^{-1}\cdot\underline{b}$

$(A^{-1}A)\underline{x} = A^{-1}\underline{b}$

$I\cdot\underline{x} = A^{-1}\underline{b}$

$\boxed{\underline{x} = A^{-1}\underline{b}}$

formel som gjelder når A er invertibel

$\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1\end{pmatrix}\cdot\begin{pmatrix} 14 \\ -17\end{pmatrix}$

$\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} -5\cdot 14 - 3\cdot 17 \\ 2\cdot 14 + 17\end{pmatrix} = \begin{pmatrix} -121 \\ 45\end{pmatrix}$

n° generell:

A $n\times n$-matrise

$|A|\neq 0$: $A^{-1} = \frac{1}{|A|}\cdot \underline{\text{adj}(A)}$

den adjungerte matrisen til A

$|A| = 0$: A^{-1} eksisterer ikke

Den adj. matrisen: $\text{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T$

Eks: $A = \begin{pmatrix} 2 & 5 & 10 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$$|A| = +\cancel{2}(6) - \cancel{5}(5) + \cancel{10}(1)$$
$$= 12 - 25 + 10 = -3 \neq 0$$

A^{-1} eksisterer (A invertibel)

$$A^{-1} = \frac{1}{-3}\cdot\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$C_{11} = 6$ $C_{12} = -5$ $C_{13} = 1$

$C_{21} = -15$ $C_{22} = 8$ $C_{23} = -1$

$C_{31} = 0$ $C_{32} = 2$ $C_{33} = -1$

$$= \frac{1}{-3}\begin{pmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 0 & 2 & -1 \end{pmatrix}^T$$

$$= \frac{1}{3}\begin{pmatrix} -6 & 5 & -1 \\ 15 & -8 & 1 \\ 0 & -2 & 1 \end{pmatrix}^T$$

$$= \frac{1}{3}\begin{pmatrix} -6 & 15 & 0 \\ 5 & -8 & -2 \\ -1 & 1 & 1 \end{pmatrix}$$

Eks:

$$2x + 5y + 10z = 1$$
$$x + 2y + 4z = 1$$
$$x + 3y + 9z = 1$$

$A\underline{x} = \underline{b}$

$\underline{x} = A^{-1}\underline{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\frac{1}{3}\begin{pmatrix} -6 & 15 & 0 \\ 5 & -8 & -2 \\ -1 & 1 & 1 \end{pmatrix}}_{A^{-1}}\cdot\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\underline{b}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5/3 \\ 1/3 \end{pmatrix}$$

Eks: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| \neq 0$

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2 Indreprodukt av vektorer

$\underline{v}, \underline{w}$ n-vektorer: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

Defn: Indreprodukt (prikkprodukt) av vektorene

$$\boxed{\underline{v} \cdot \underline{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n}$$

Merk:

i) $\underline{v} \cdot \underline{w}$ er et tall

ii) $\underline{v} \cdot \underline{v} = v_1^2 + v_2^2 + \dots + v_n^2 \geq 0$
$= \|\underline{v}\|^2$

(Husk: $\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$)

iii) $\underline{v} \cdot \underline{w} = 0 \iff \underline{v} \perp \underline{w}$
(vektorene $\underline{v}, \underline{w}$ er ortogonale, eller står 90° på hverandre)

iv) Likner på utregning i matrisemult.

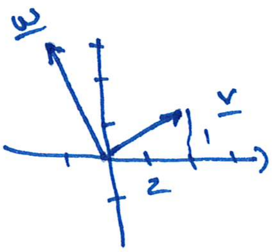
Eks:

$\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 2 \cdot (-1) + 1 \cdot 3 = -2 + 3 = \underline{\underline{1}}$

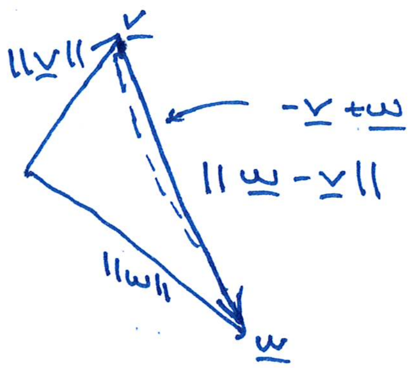
$\underline{v} \cdot \underline{v} = 2^2 + 1^2 = 5$

$\|\underline{v}\| = \sqrt{5}$



Forklaring: $\underline{v} \perp \underline{w} \iff \underline{v} \cdot \underline{w} = 0$

(ortogonale, eller 90°)



$$\|\underline{v}\|^2 + \|\underline{w}\|^2 = v_1^2 + v_2^2 + \dots + v_n^2 + w_1^2 + w_2^2 + \dots + w_n^2$$

$$\|\underline{w} - \underline{v}\|^2 = (w_1 - v_1)^2 + (w_2 - v_2)^2 + \dots + (w_n - v_n)^2$$

$$\underline{w} - \underline{v} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} w_1 - v_1 \\ w_2 - v_2 \\ \vdots \\ w_n - v_n \end{pmatrix}$$

$$\|\underline{w} - \underline{v}\|^2 = (\underline{w} - \underline{v}) \cdot (\underline{w} - \underline{v})$$

Altså: $\underline{v} \perp \underline{w} \iff \cancel{v_1^2} + \cancel{v_2^2} + \dots + \cancel{v_n^2} + \cancel{w_1^2} + \cancel{w_2^2} + \dots + \cancel{w_n^2}$

$$= \cancel{w_1^2} - 2w_1v_1 + \cancel{v_1^2} + \cancel{w_2^2} - 2w_2v_2 + \cancel{v_2^2} + \dots + \cancel{w_n^2} - 2w_nv_n + \cancel{v_n^2}$$

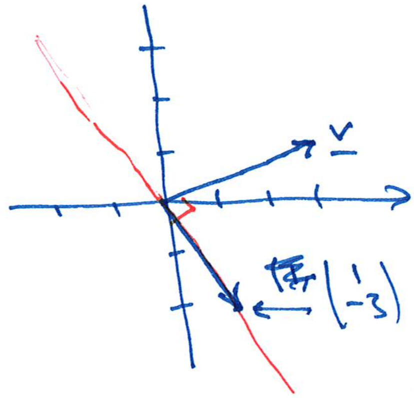
$$:(-2) \,|\; 0 = -2v_1w_1 - 2v_2w_2 - \dots - 2v_nw_n$$

$$0 = v_1w_1 + v_2w_2 + \dots + v_nw_n$$

$$\boxed{\underline{v} \cdot \underline{w} = 0}$$

Eks: $\underline{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Hvilke vektorer står normalt på $\underline{v}$



Svar: $\underline{w} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\underline{w} \perp \underline{v} \iff \underline{w} \cdot \underline{v} = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3x + y = 0$$

$$y = -3x, \ x \text{ fri}$$

$$\underline{w} = \begin{pmatrix} x \\ y \end{pmatrix} = \cancel{\begin{pmatrix} -3x \\ x \end{pmatrix} = x \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} x \\ -3x \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

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