MET1180, forelesning 27, 16. jan. 2024, Runar Ile

Plan
1. Repetisjon av ubestemte integraler
2. Integrasjon av rasjonale uttrykk

1. Repetisjon

Integrasjonsregler

- Potensregelen $\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C \quad (n \neq -1)$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- Addisjonsregelen: $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- Konstantregelen: $\int c\cdot f(x)\,dx = c\cdot\int f(x)\,dx$ (c konst.)
- $\int e^x\,dx = e^x + C$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$ (a konst.)

Oppg 5b
$$\int \frac{x^3+2x-2}{x}\,dx = \int \frac{x^3}{x} + \frac{2x}{x} - \frac{2}{x}\,dx$$
$$= \int x^2\,dx + \int 2\,dx - \int \frac{2}{x}\,dx$$
$$= \frac{1}{3}x^3 + 2x - 2\ln|x| + C$$

Oppg 5d
$$\int \frac{\sqrt{x}+1}{x^2}\,dx \overset{\text{enkel nevner}}{=} \int \frac{\sqrt{x}}{x^2}\,dx + \int \frac{1}{x^2}\,dx$$
$$= \int \underbrace{x^{\frac{1}{2}}\cdot x^{-2}}_{x^{\frac{1}{2}-2} = x^{-1{,}5}}\,dx + \int x^{-2}\,dx = \frac{1}{-0{,}5}\cdot x^{-0{,}5} + \frac{1}{-1}\cdot x^{-1} + C$$
$$= \frac{-2}{\sqrt{x}} - \frac{1}{x} + C$$

Delvis integrasjon

$$\int u \cdot v' = u \cdot v - \int u' \cdot v$$

Oppg 10b $\int x \cdot e^{-x}\,dx$

Setter $u = x$ og $v' = e^{-x}$

så $u' = 1$ $\quad v = -e^{-x}$

$$\overset{\text{delvis int.}}{=} -xe^{-x} - \int -e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C = \underline{\underline{-(x+1)e^{-x} + C}}$$

Oppg 9e $\int \frac{\ln(x)}{x^2}\,dx = \int \ln(x) \cdot x^{-2}\,dx$

setter $u = \ln(x)$ og $v' = x^{-2}$

og $u' = \frac{1}{x}$ $\quad v = -x^{-1}$

$$\overset{\text{delvis int.}}{=} \ln(x) \cdot (-x^{-1}) - \int \underbrace{\frac{1}{x} \cdot (-x^{-1})}_{=x^{-1}}\,dx + C$$

$$= -\frac{1}{x} \cdot \ln(x) + \int x^{-2}\,dx = -\frac{1}{x}\ln(x) - x^{-1} + C$$

$$= -\frac{1}{x} \cdot \ln(x) - \frac{1}{x} + C = \underline{\underline{-\frac{1}{x}\left(\ln(x) + 1\right) + C}}$$

Oppg 8a $\int \ln(x)\,dx = \int \ln(x) \cdot 1\,dx$

Setter $u = \ln(x)$ og $v' = 1$

så $u' = \frac{1}{x}$ $\quad v = x$

$$\overset{\text{delvis int.}}{=} \ln(x) \cdot x - \int \overbrace{\frac{1}{x} \cdot x}^{1}\,dx$$

$$= \ln(x) \cdot x - x + C$$

$$= \underline{\underline{x \cdot \ln(x) - x + C}}$$

<u>Substitusjon</u> $\int g(u(x))\,dx = \int \underbrace{g(u) \cdot \frac{1}{u'(x)}}\,du$

- ingen x-er!

<u>Oppg 5c</u> $\int \frac{6x}{1+3x^2}\,dx$

Setter $u = 1+3x^2$

$du = (1+3x^2)' \cdot dx$

$= 6x \cdot dx \quad |:6x$

så $dx = \frac{1}{6x}\,du$

$$= \int \frac{\cancel{6x}}{u} \cdot \frac{1}{\cancel{6x}}\,du$$

$$= \int \frac{1}{u}\,du$$ [NB: Ingen x-er!]

$$= \ln|u| + C = \ln|1+3x^2| + C$$

$$= \underline{\underline{\ln(1+3x^2) + C}}$$

Start: 11.00

<u>Oppg 11</u> $\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx$

Setter $u = 1-\sqrt{x}$

$du = (1-\sqrt{x})' \cdot dx$

$= -(x^{\frac{1}{2}})' \cdot dx$

$= -\frac{1}{2} x^{-\frac{1}{2}} \cdot dx$

$= -\frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} \cdot dx$

$= -\frac{1}{2\sqrt{x}} \cdot dx \quad |\cdot(-2)$

så $-2\,du = \frac{1}{\sqrt{x}} \cdot dx$

subst.

$$= -2 \int e^u\,du$$

$$= -2e^u + C$$

$$= \underline{\underline{-2e^{1-\sqrt{x}} + C}}$$

Vanskeligst: Når er det hva?

Oppg 7a $\int x\sqrt{x^2+1}\,dx$

– substitusjon! (ikke delvis int.)

Setter $u = x^2+1$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

delvis int.

$$= \int \cancel{x}\sqrt{u}\cdot\frac{1}{2\cancel{x}}\,du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot u^{\frac{3}{2}} + C$$

$$= \underline{\underline{\frac{1}{3}(x^2+1)^{\frac{3}{2}} + C}} = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + C$$

Oppg 7c $\int x e^{-x^2}\,dx$

Delvis virker ikke!

Setter $u = -x^2$

$du = -2x\cdot dx$

$dx = \frac{1}{-2x}\cdot du$

subst

$$= \int \cancel{x}\cdot e^{u}\cdot\frac{1}{-2\cancel{x}}\cdot du$$

$$= -\frac{1}{2}\int e^{u}\,du = -\frac{1}{2}e^{u} + C = \underline{\underline{-\frac{1}{2}e^{-x^2} + C}}$$

Eks $\int 3x\ln(x^2+5)\,dx$

Substitusjon:

$u = x^2+5$

$du = 2x\cdot dx$

$dx = \frac{1}{2x}\cdot du$

$$= \int 3\cancel{x}\cdot\ln(u)\cdot\frac{1}{2\cancel{x}}\cdot du$$

$$= \frac{3}{2}\int \ln(u)\,du = \frac{3}{2}\left(u\ln(u) - u\right) + C$$

$$= \frac{3}{2}\left[(x^2+5)\ln(x^2+5) - (x^2+5)\right] + C$$

$$= \underline{\underline{\frac{3}{2}(x^2+5)\left[\ln(x^2+5) - 1\right] + C}}$$

Men her kan vi også bruke delvis int:

$u = \ln(x^2+5)$ og $v' = 3x$

$u' = \dfrac{2x}{x^2+5}$ $\qquad v = \frac{3}{2}x^2$

$$\int 3x \ln(x^2+5)\,dx = \frac{3}{2}x^2 \ln(x^2+5) - \int \underbrace{\frac{\cancel{2}x}{x^2+5} \cdot \frac{3}{\cancel{2}} \cdot x^2}\,dx$$

$\dfrac{3x^3}{x^2+5} \overset{\text{polyn. div}}{=} 3x - \dfrac{15x}{x^2+5}$

og $\displaystyle\int 3x - \frac{15x}{x^2+5}\,dx$

$= \frac{3}{2}x^2 - \frac{15}{2}\ln(x^2+5) + C$

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2. Integrasjon av rasjonale uttrykk

Rasjonalt uttrykk $\dfrac{p(x)}{q(x)}$ ← polynomer

Eks $\displaystyle\int \frac{1}{3x-5}\,dx = \int \frac{1}{u}\cdot\frac{1}{3}\,du = \frac{1}{3}\ln|u| + C$

$= \underline{\underline{\frac{1}{3}\ln|3x-5| + C}}$

subst:
$u = 3x-5$
$du = 3\,dx$
$dx = \frac{1}{3}du$

Eks $\displaystyle\int \frac{x^2}{1-x}\,dx$

Hvis grad (teller) $\geq$ grad (nevner) så polynomdividerer vi.

$$x^2 : (-x+1) = -x-1+\frac{1}{-x+1}$$
$$\underline{-(x^2-x)}$$
$$x$$
$$\underline{-(x-1)}$$
$$1$$

$$= \int -x-1+\frac{1}{1-x}\,dx$$

$$= \int -x\,dx + \int -1\,dx + \int \frac{1}{1-x}\,dx$$

$$= -\frac{1}{2}x^2 - x - \ln|1-x| + C$$

Eks $\displaystyle\int \frac{3}{(x+5)^2}\,dx = \int \frac{3}{u^2}\,du$

subst:

$u = x+5$

$du = dx$

$$= 3\int u^{-2}\,du = 3\cdot\frac{1}{-1}u^{-1} + C$$

$$= -\frac{3}{u} + C = -\frac{3}{x+5} + C$$