| | Emne | Lærebok | Oppgaver |
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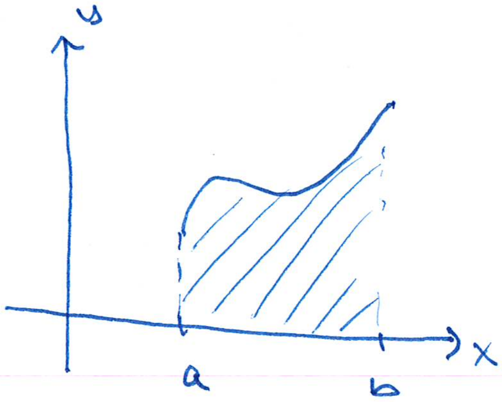
| 1 | Bestemte integral som areal | [E] 5.6 | 5.6.4 - 5.6.5 |
| 2 | Økonomiske anvendelser av integrasjon | [E] 5.7 | 5.7.1 - 5.7.6 |

## 1 Bestemt integral som areal




A = arealet under grafen til f i intervallet [a,b]

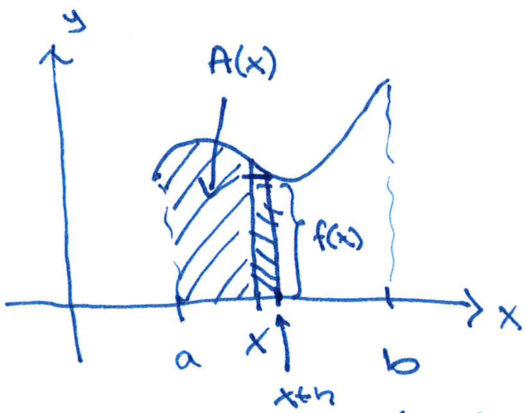
Antar: - f kontinuerlig
- $f(x) \geq 0$ i $[a,b]$

Teorem: $\boxed{A = \int_a^b f(x)\,dx}$

$= [F(x)]_a^b = F(b) - F(a)$

(der $F'(x) = f(x)$)

Forklaring ("bevis")




$a \leq x \leq b$:

Arealfunksjon $A(x) =$ ~~integ~~ arealet under grafen til f i intervallet $[a,x]$

Vi ser at: $A(a) = 0$

$A(b) = A$

Kan vise: $A'(x) = f(x)$

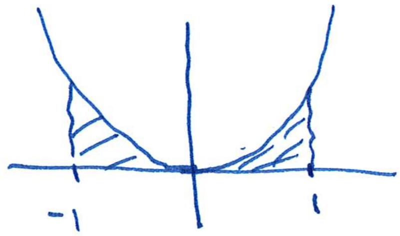
Det betyr: $\int_a^b f(x)dx = [A(x)]_a^b = A(b) - A(a) = A$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$\simeq \frac{A(x+h) - A(x)}{h}$$ når h er liten

$$= \frac{\text{arealet av stripen}}{h} \simeq \frac{f(x) \cdot \cancel{h}}{\cancel{h}} = f(x)$$

**Ek:** $A =$ arealet under $f(x)=x^2$ i intervallet $[-1,1]$

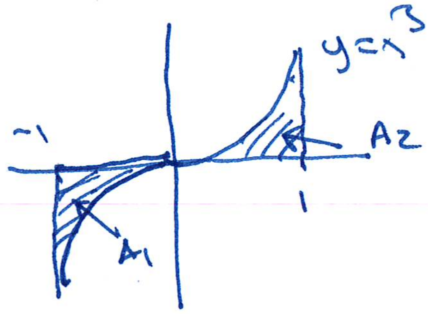



$$A = \int_{-1}^{1} x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{-1}^{1} = \left(\tfrac{1}{3}\cdot 1^3\right) - \left(\tfrac{1}{3}\cdot(-1)^3\right)$$

$$= 1/3 + 1/3 = \underline{\underline{2/3}}$$

$f(x)=x^2$ er symm. om y-aksen $\longrightarrow$ $\| \quad 2\cdot\int_0^1 x^2\,dx$

**Ek:** $A =$ arealet under $f(x)=x^3$ i $[-1,1]$




$$\int_{-1}^{1} x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{-1}^{1}$$

$$= \left(\tfrac{1}{4}\cdot 1^4\right) - \left(\tfrac{1}{4}\cdot(-1)^4\right) = \underline{0}$$

$$\int_0^1 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_0^1 = \underline{1/4}$$

$$\int_{-1}^{0} x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{-1}^{0} = 0 - \left(\tfrac{1}{4}(-1)^4\right) = \underline{-1/4}$$

**Fakta:**

Når $f(x) \leq 0$ for alle $x$ i $[a,b]$ så er $\int_a^b f(x)\,dx = -A \iff A = -\int_a^b f(x)\,dx$

der $A$ er arealet mellom x-aksen og grafen.

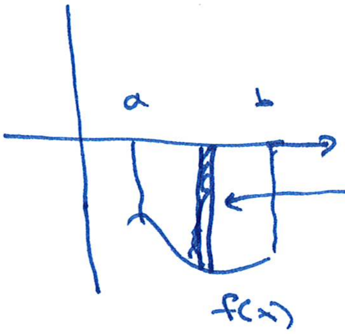
**Fakta:** $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$$0 = \int_{-1}^{1} x^3\,dx = \underbrace{\int_{-1}^{0} x^3\,dx}_{-A_1} + \underbrace{\int_0^1 x^3\,dx}_{A_2}$$

$$\Downarrow$$

$$A_1 + A_2 = -\underbrace{\int_{-1}^{0} x^3\,dx}_{A_1} + \underbrace{\int_0^1 x^3\,dx}_{A_2}$$

## Forklaring:



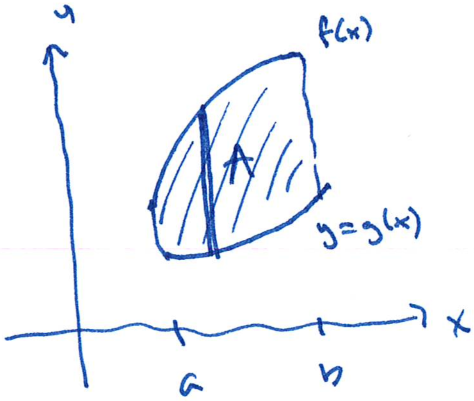

Areal $= \Delta x \cdot (0 - f(x)) = \Delta x \cdot (-f(x))$

$\Rightarrow \underline{A'(x) = -f(x)}$

**Faktum:**

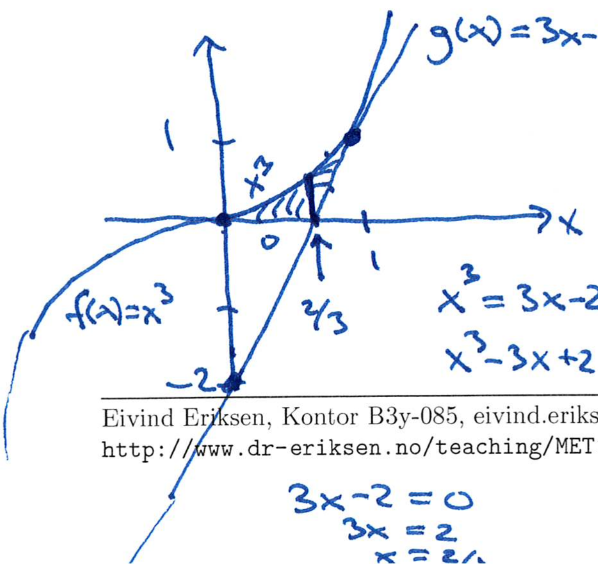
Når $f(x) \geq g(x)$ i intervallet $[a,b]$ så er arealet av området mellom $f(x)$ og grafen til $g(x)$ i intervallet gitt ved:

$$A = \int_a^b f(x) - g(x)\, dx$$




Tolkning: summen av arealet av tynne striper m/ høyde $f(x)-g(x)$ og bredde $dx$ i intervallet fra a til b.

**Eks:** Finn arealet av området i første kvadrant begrenset av $f(x)=x^3$, $g(x)=3x-2$ og x-aksen.




$x^3 = 3x-2$

$x^3-3x+2=0$

$(x-1)(x-1)(x+2)=0$

$(x-1)(x^2+x-2)=0$

$x^3-3x+2 : x-1 = x^2+x-2$

$-(x^3-x^2)$

$x^2-3x+2$

$x^2-x$

$3x-2=0$

$3x=2$

$x=2/3$

$$A = \int_0^{2/3} x^3\,dx + \int_{2/3}^{1} x^3-(3x-2)\,dx$$

$$= \left[\tfrac14 x^4\right]_0^{2/3} + \left[\tfrac14 x^4 - \tfrac32 x^2 + 2x\right]_{2/3}^{1}$$

$$= \cancel{\tfrac14\left(\tfrac23\right)^4} - 0 + \left(\tfrac14\cdot 1^4 - \tfrac32\cdot 1^2 + 2\cdot 1\right) - \left(\cancel{\tfrac14\cdot\left(\tfrac23\right)^4} - \tfrac32\cdot\left(\tfrac23\right)^2 + 2\cdot\tfrac23\right)$$

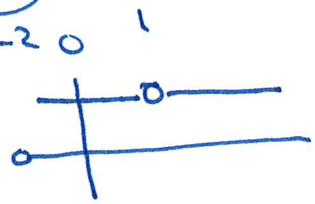
$$= \tfrac14 - \tfrac32 + 2 + \tfrac23 - \tfrac43$$

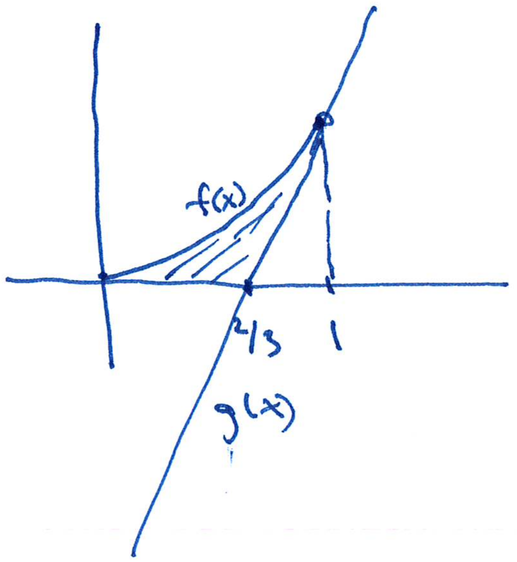
$$= 3/4 - 2/3 = 1/12$$

Tenker at $x^3 \geq 3x-2$ i $[0,1]$ ok

$$x^3-3x+2 \geq 0$$

$$(x-1)^2(x+2) \geq 0$$




$$A = \int_0^1 \underset{x^3}{\underset{\shortparallel}{f(x)}}\,dx - \int_{2/3}^1 \underset{3x-2}{\underset{\shortparallel}{g(x)}}\,dx$$

## 2 Anvendelser

### i) Sannsynlighetsregning

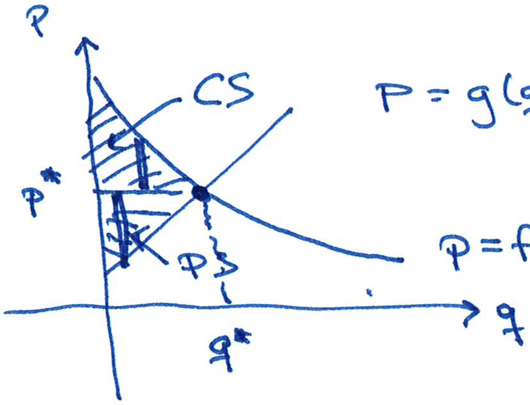

f(x) = tetthets-funksjonen til X

$$P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$$

Eks: normalfordeling

X normalfordelt

## ii) Konsument/produsent overskudd

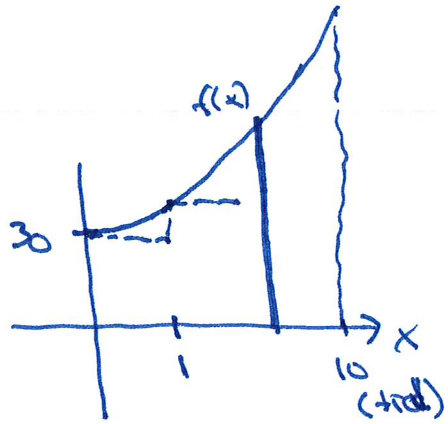



$p = g(q)$ (omvendt) tilbudsfn.

$p = f(q)$ (omvendt) etterspørselfn.

$$CS = \int_0^{q^*} f(q) - p^* \, dq \qquad PS = \int_0^{q^*} p^* - g(q) \, dq$$

## iii) Kontantstrømmer




Eks: Leieinntekt på 30 MNOK/år som øker med 6% per år. Samlet leieinntekt i.l.a. 10 år

$$\int_0^{10} 30 \cdot 1.06^x \, dx \qquad f(x) = 30 \cdot 1.06^x$$

$$= 30 \left[ \frac{1}{\ln(1.06)} e^{\ln(1.06)x} \right]_0^{10}$$

$$= 30 \left( \frac{1}{\ln 1.06} \underbrace{e^{\ln(1.06) \cdot 10}}_{1.06} \right) - 30 \left( \frac{1}{\ln 1.06} \cdot 1 \right)$$

$$= \frac{30}{\ln 1.06} \left( 1.06^{10} - 1 \right)$$

Husk:

$$1.06^x = \left(e^{\ln 1.06}\right)^x = e^{\ln(1.06) \cdot x}$$

$$\int 1.06^x \, dx = \int e^{\ln(1.06) \cdot x} \, dx = \frac{1}{\ln 1.06} e^{\ln(1.06)x} + C$$

Samlet nåverdi:

kont. diskontering, diskonteringsrente. $= r$

$r = 10\%$:

$$\int_0^{10} f(x) \cdot e^{-rx} \, dx = \int_0^{10} 30 \cdot 1.06^x \, e^{-0.10x} \, dx$$

$$= \int_0^{10} 30 \, \underbrace{e^{\ln 1.06 \cdot x} \cdot e^{-0.10x}}_{e^{(\ln 1.06 - 0.10)x}} \, dx$$