	Emne	Lærebok	Oppgaver
1	Repetisjon og oppgaveregning		
2	Inverse matriser	[E] 6.7	[E] 6.7.1 - 6.7.4

① Repetisjon:

- matrisemultiplikasjon $(AB \neq BA)$
- transponering
- identitetsmatrisen $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\boxed{\begin{aligned} A \cdot I &= A \\ I \cdot A &= A \end{aligned}}$$

* Formler: $|AB| = |A| \cdot |B|$

$(AB)^T = B^T A^T$

Oppgaveark 37

7.) $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

a) $AX = I$

Alt: $AX = I \mid A^{-1} \cdot$ $A^{-1}AX = A^{-1}I$

$X = A^{-1}$

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x+3z & y+3w \\ 2x+5z & 2y+5w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} x \quad + 3z \quad &= 1 \\ y \quad + 3w &= 0 \\ 2x \quad + 5z \quad &= 0 \\ 2y \quad + 5w &= 1 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 2 & 0 & 5 & 0 & 0 \\ 0 & 2 & 0 & 5 & 1 \end{array}\right) \begin{matrix} \\ \\ \\ \end{matrix} -2 \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 2 & 0 & 5 & 1 \end{array}\right) -2 \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right)$$

$$-w = +1 \Rightarrow \underline{w = -1} \qquad y + 3(-1) = 0 \qquad \underline{y = 3}$$

$$-z = -2 \qquad \underline{z = 2} \qquad x + 3(2) = 1 \qquad \underline{x = -5}$$

$$X = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$$

b) $AX = XA$

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} x+3z & y+3w \\ 2x+5z & 2y+5w \end{pmatrix} = \begin{pmatrix} x+2y & 3x+5y \\ z+2w & 3z+5w \end{pmatrix}$$

$$\begin{aligned} \cancel{x} + 3z &= \cancel{x} + 2y \\ y + 3w &= 3x + 5y \\ 2x + 5z &= z + 2w \\ 2y + \cancel{5w} &= 3z + \cancel{5w} \end{aligned} \qquad \begin{aligned} -2y + 3z &= 0 \ \checkmark \\ -3x - 4y + 3w &= 0 \ \checkmark \\ 2x + 4z - 2w &= 0 \ \checkmark \\ 2y - 3z &= 0 \ \checkmark \end{aligned}$$

$$\left(\begin{array}{cccc|c} 2 & 0 & 4 & -2 & 0 \\ -3 & -4 & 0 & 3 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 2 & -3 & 0 & 0 \end{array}\right) \begin{array}{l} :2 \\ \\ \\ ]1 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ -3 & -4 & 0 & 3 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)]3$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & -4 & 6 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ :2 \ ]-1 \\ \\ \\ \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

z, w fri

$$-2y + 3z = 0 \qquad \frac{-2y}{-2} = \frac{-3z}{-2} \qquad y = \underline{3z/2}$$

$$x + 2z - w = 0 \qquad x = \underline{-2z + w}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z+w \\ 3z/2 \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z \\ 3z/2 \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} w \\ 0 \\ 0 \\ w \end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = z\cdot\begin{pmatrix}-2\\3/2\\1\\0\end{pmatrix} + w\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$X = z\cdot\begin{pmatrix}-2 & 3/2\\1 & 0\end{pmatrix} + w\cdot\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$= \frac{z}{2}\begin{pmatrix}-4 & 3\\2 & 0\end{pmatrix} + w\underbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}_{I}$$

uendelig mange løsn.

c) $X^2 = A$

$$\begin{pmatrix}x & y\\z & w\end{pmatrix}\cdot\begin{pmatrix}x & y\\z & w\end{pmatrix} = \begin{pmatrix}1 & 3\\2 & 5\end{pmatrix}$$

$x^2 + yz = 1 \qquad yz = 1 - x^2$

$xy + zw = 3$

$xz + zw = 2$

$yz + w^2 = 5 \quad \Rightarrow \quad 1 - x^2 + w^2 = 5$

$w^2 - x^2 = 5$

$\underline{w^2 = x^2 + 5}$

ser vanskelig ut

$X^2 = A \qquad$ " $X = \sqrt{A}$ "

$|X^2| = |A|$

$\parallel$

$|X\cdot X|$

$|X|\cdot|X| = |A|$

$|X|^2 = |A| = 1\cdot 5 - 3\cdot 2 = -1$

ikke mulig $\Rightarrow$ ingen løsn av $X^2 = A$

8. a) $a = 1$

$$\left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\1 & 2 & 3 & -1\\1 & 3 & 0 & 2\end{array}\right) \begin{array}{l} \\ ]\cdot(-1) \\ ]\cdot(-1)\end{array} \rightarrow \left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\0 & 1 & 2 & -2\\0 & 2 & -1 & 1\end{array}\right) \begin{array}{l} \\ \\ ]\cdot(-2)\end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c}1 & 1 & 1 & 1\\0 & 1 & 2 & -2\\0 & 0 & -5 & 5\end{array}\right)$$

trappeform

$x + y + z = 1 \qquad x + 0 + (-1) = 1 \qquad \underline{x = 2}$

$y + 2z = -2 \qquad y + 2(-1) = -2 \qquad \underline{y = 0}$

$-5z = 5 \qquad \underline{z = -1}$

En entydig løsn. $(x, y, z) = \underline{\underline{(2, 0, -1)}}$

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b) $$|A| = \begin{vmatrix} a & 1 & a \\ 1 & 2 & 3 \\ a & 3 & 0 \end{vmatrix} = a(0-9) - 1(0-3a) + a(3-2a)$$

$$= -9a + 3a + 3a - 2a^2$$

$$= \underline{\underline{-2a^2 - 3a}} = -a(2a+3)$$

$$= -2a\left(a + 3/2\right)$$

$\underline{|A|=0}$: $-a(2a+3) = 0$

$\underline{\underline{a=0}}$ eller $2a+3=0$

$a = \underline{\underline{-3/2}}$

c) Uendelig mange løsn:

Eneste muligheter: $\underline{a=0}$, $a=\underline{-3/2}$ ved $|A|=0$.

$|A| \neq 0$: én entydig løsn
$|A| = 0$: ingen eller uendelig mange løsn.

$\underline{a=0}$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 3 \end{array}\right) \cdot(-3) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

uendelig mange løsn.

trappeform

$a = \underline{-3/2}$:

$$\left(\begin{array}{ccc|c} -3/2 & 1 & -3/2 & 1 \\ 1 & 2 & 3 & 3/2 \\ -3/2 & 3 & 0 & 3/2 \end{array}\right) \begin{array}{l} \cdot 2 \\ \cdot 2 \\ \cdot 2 \end{array} \to \left(\begin{array}{ccc|c} -3 & 2 & -3 & 2 \\ 2 & 4 & 6 & 3 \\ -3 & 6 & 0 & 3 \end{array}\right) \cdot 1 \to \left(\begin{array}{ccc|c} -1 & 6 & 3 & 5 \\ 2 & 4 & 6 & 3 \\ -3 & 6 & 0 & 3 \end{array}\right) \begin{array}{l} \cdot 2 \\ \cdot(-3) \end{array}$$

$$\to \left(\begin{array}{ccc|c} -1 & 6 & 3 & 5 \\ 0 & 16 & 12 & 13 \\ 0 & -12 & -9 & -12 \end{array}\right) \quad \frac{12}{16} = \frac{3}{4} \quad \to \left(\begin{array}{ccc|c} -1 & 6 & 3 & 5 \\ 0 & 16 & 12 & 13 \\ 0 & 0 & 0 & -12 + 13 \cdot 3/4 \end{array}\right)$$

$$-12 + 13 \cdot 3/4 \neq 0$$

Ingen løsn.

Konkl: uendelig mange løsn. for $\underline{\underline{a = 0}}$

d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$ når $a = 1$.

$$A^2 - 3A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix} - 3 \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 & 4 \\ 6 & 14 & 7 \\ 4 & 7 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 3 \\ 3 & 6 & 9 \\ 3 & 9 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 3 & 1 \\ 3 & 8 & -2 \\ 1 & -2 & 10 \end{pmatrix}}}$$

② Inverse matriser

A matrise

Defn En invers matrise for A er en matrise B slik at $\underbrace{AB = I \text{ og } BA = I}$.

$A \cdot A^{-1} = I$

$A^{-1} \cdot A = I$

Fakta:
- Ikke alle matriser har en invers. A kalles invertibel hvis A har en invers matrise
- hvis A har en invers matrise, så er den entydig og vi kaller den A^{-1}

Eks:

$$x + y + z = 13$$
$$x - y + z = 25$$
$$x + 2y + 4z = 17$$

$3x = 7 \quad | \cdot \frac{1}{3} = 3^{-1}$

$\cancel{\frac{1}{3}} \cancel{3}x = \frac{1}{3} \cdot 7$

Matriseform

$A \cdot \underline{x} = \underline{b} \quad | A^{-1} \cdot$

$A^{-1} A \underline{x} = A^{-1} \underline{b}$

$I \underline{x} = A^{-1} \underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 13 \\ 25 \\ 7 \end{pmatrix}$$

Anta at A har en invers A^{-1}.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{A^{-1}} \cdot \begin{pmatrix} 13 \\ 25 \\ 17 \end{pmatrix}$$

én entydig løsn. hvis A^{-1} fins

Resultat:
En matrise A er invertibel (A^{-1} fins) $\Leftrightarrow$ A er kvadratisk og $|A| \neq 0$.

Eks: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2×2-matriser

$|A| = ad - bc$:

$ad - bc \neq 0$: A^{-1} fins

$ad - bc = 0$: A^{-1} fins ikke

og

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Eks: $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$

$$|A| = 1 \cdot 5 - 2 \cdot 3 = -1 \neq 0$$

$$A^{-1} = \frac{1}{-1} \cdot \begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$$

$$A^{-1} = \underline{\underline{\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}}}$$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Generell formel for A^{-1}:

A $n \times n$-matrise, $|A| \neq 0 \Rightarrow A^{-1} = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T}_{\text{adj}(A)}$

Eks:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

$C_{11} = +(-4-2) = \underline{-6}$ $\quad C_{12} = -(4-1) = \underline{-3}$ $\quad C_{13} = +(2-(-1)) = \underline{3}$

$C_{21} = -(4-2) = -2$ $\quad C_{22} = +(4-1) = \underline{3}$ $\quad C_{23} = -(2-1) = \underline{-1}$

$C_{31} = +(1+1) = 2$ $\quad C_{32} = -(1-1) = \underline{0}$ $\quad C_{33} = +(-1-1) = \underline{-2}$

$|A| = 1 \cdot (-6) + 1 \cdot (-3) + 1 \cdot 3 = \underline{-6} \neq 0.$

A^{-1} fins

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{-6} \begin{pmatrix} -6 & -3 & 3 \\ -2 & 3 & -1 \\ 2 & 0 & -2 \end{pmatrix}^T = \frac{1}{-6} \begin{pmatrix} -6 & -2 & 2 \\ -3 & 3 & 0 \\ 3 & -1 & -2 \end{pmatrix} = \underline{\underline{\frac{1}{6} \begin{pmatrix} 6 & 2 & -2 \\ 3 & -3 & 0 \\ -3 & 1 & 2 \end{pmatrix}}}$$

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