	Emne	Lærebok	Oppgaver
1	Mer om inverse matriser	[E] 6.7	
2	Oppgaveregning: Eksamen 05/2023		Oppgave 1, 3

Se eksamenssiden for løsn. av Oppg. 3

① Inverse matriser   Rep + fortsettelse

A
$n \times n$-matrise

Defn: Matrisen har en invers A^{-1} hvis det fins en matrise A^{-1} slik at

$$\boxed{A \cdot A^{-1} = I \quad \text{og} \quad A^{-1} \cdot A = I}$$

A kalles invertibel hvis A^{-1} fins

Husk:
$\det(A)$, A^{-1},
A^n $n = 2, 3, \ldots$
kun når A er kvadratisk

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
identitetsmatrisen

Fakta: (i) A er invertibel $\Leftrightarrow$ $|A| \neq 0$
(A^{-1} fins)

(ii) Hvis A er invertibel, så er A^{-1} entydig

Forklaring

Anta B og C er to matriser som er inverser til A: $AB = BA = I$
$AC = CA = I$

Da har vi: $BAC = (BA)C = IC = C$
$= B(AC) = B \cdot I = B$

$\Rightarrow B = C$

Eks:

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

i) Er A invertibel? Ja A^{-1} fins.

$|A| = 2(4-1) - 1(2-1) + 1(1-2) = 6 - 1 - 1 = 4 \neq 0$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$|A| = +2(3) - 1(1) + 1\cdot(-1) = 4 \neq 0$$

$$\begin{pmatrix} C_{11} = +3 = \underline{3} & C_{12} = -1 = \underline{-1} & C_{13} = +(-1) = \underline{-1} \\ C_{21} = \underline{-1} & C_{22} = \underline{3} & C_{23} = \underline{-1} \\ C_{31} = \underline{-1} & C_{32} = \underline{-1} & C_{33} = \underline{3} \end{pmatrix}$$

ii) Bestem A^{-1}:

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}^T = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix}$$

Eks:

$$\begin{aligned} 2x + y + z &= 4 \\ x + 2y + z &= 2 \\ x + y + 2z &= -1 \end{aligned}$$

Matriseform

$$A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b}$$

$$\cancel{A^{-1}A}\,\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 11 \\ 3 \\ -9 \end{pmatrix}$$

$$(x, y, z) = (11/4, 3/4, -9/4)$$

Metode I: A^{-1} vha kofaktorer

Hvis $|A| \neq 0$, så er

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\text{kofaktor-matrisen til } A}{}^T$$

den adjungerte matrisen til A

Spesialtilfelle: $n = 2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = ad - bc \neq 0 \Rightarrow A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Metode 2: A^{-1} vha Gauss-Jordan eliminasjon

A $n\times n$-matrise

i) Sett opp matrisen $(A\,|\,I)$

ii) Finn den reduserte trappeformen

$$(A\,|\,I) \xrightarrow{\text{regner og regner...}} \dots \to \underbrace{(B\,|\,C)}_{\text{redusert trappeform}}$$

$B = I \iff |A| \neq 0$

iii) Konklusjon: $\begin{cases} B = I: & A^{-1} = C \\ B \neq I: & A \text{ er ikke invertibel} \end{cases}$

Eks:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \begin{matrix} -1 \\ -2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 1 & 0 & -2 \end{array}\right) 1 \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -4 & 1 & 1 & -3 \end{array}\right) :-4$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right) \begin{matrix} 1 \\ -2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/4 & 2/4 & -2/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right) -1$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right)$$

Konklusjon: $A^{-1} = \begin{pmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$ //

Nyttig formel: $\boxed{(AB)^{-1} = B^{-1}\cdot A^{-1}}$ (hvis A, B er invertible og av samme størrelse)

$$(AB)(B^{-1}A^{-1}) = A\cancel{B}\cancel{B^{-1}}A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\cancel{A^{-1}}\cancel{A}B = B^{-1}B = I$$

Forklaring

2 Eksamen 05/2023

1. $A = \begin{array}{c} \begin{pmatrix} 1 & -1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 2 \end{pmatrix} \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array}$ $\qquad \underline{b} = \begin{pmatrix} 11 \\ 2 \\ 0 \end{pmatrix}$

a) Løs $A\underline{x} = \underline{b}$:

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 2 & 1 & 2 & 0 \end{array}\right) \begin{array}{l} \\ \cdot(-2) \\ \cdot(-4) \end{array} \to \left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 6 & -11 & -14 & -44 \end{array}\right) \begin{array}{l} \\ \\ \cdot(-2) \end{array}$$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 0 & -1 & 2 & -4 \end{array}\right)$$

trappeform

$$\begin{aligned} x - y + 3z + 4w &= 11 \\ 3y - 5z - 8w &= -20 \\ -z + 2w &= -4 \end{aligned}$$

w er en fri variabel

uendelig mange løsn. m/ én frihetsgrad

(3) $\dfrac{-z}{-1} = \dfrac{-4-2w}{-1}$

$z = \underline{4 + 2w}$

(2) $3y = -20 + 5(4+2w) + 8w$

$\dfrac{3y}{3} = \dfrac{18w}{3} \qquad y = \underline{6w}$

(1) $x = 11 + (6w) - 3(4+2w) - 4w$

$x = \underline{-1-4w}$

Konkl: w fri

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1-4w \\ 6w \\ 4+2w \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -4w \\ 6w \\ 2w \\ w \end{pmatrix} = \underline{\begin{pmatrix} -1 \\ 0 \\ 4 \\ 0 \end{pmatrix} + w\begin{pmatrix} -4 \\ 6 \\ 2 \\ 1 \end{pmatrix}}$$

(b) $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$ = kolonne vektorene i A

Likning: $\boxed{\underline{v_3} = a\cdot\underline{v_1} + b\underline{v_2} + c\cdot\underline{v_4}}$ ← ønsker å løse denne vektor-likningen

Alt I: $\begin{pmatrix}3\\1\\1\end{pmatrix} = a\cdot\begin{pmatrix}1\\2\\4\end{pmatrix} + b\begin{pmatrix}-1\\1\\2\end{pmatrix} + c\begin{pmatrix}4\\0\\2\end{pmatrix}$

Løs vha Gauss.

$$\left(\begin{array}{ccc|c}1 & -1 & 4 & 3\\ 2 & 1 & 0 & 1\\ 4 & 2 & 2 & 1\end{array}\right) \to \left(\begin{array}{ccc|c}1 & -1 & 4 & 3\\ 0 & 3 & -8 & -5\\ 0 & 6 & -14 & -11\end{array}\right)$$

$$\to \left(\begin{array}{ccc|c}1 & -1 & 4 & 3\\ 0 & 3 & -8 & -5\\ 0 & 0 & 2 & -1\end{array}\right)$$

Ja:
$$a - b + 4c = 3$$
$$3b - 8c = -5$$
$$2c = -1$$

$c = -1/2$

$3b = -5 + 8(-1/2) = -9 \quad b = -3$

$a = 3 + (-3) - 4(-1/2) = 2$

Alt 2: $\underline{v_3} = a\underline{v_1} + b\underline{v_2} + c\underline{v_4}$

$$a\underline{v_1} + b\underline{v_2} - 1\cdot\underline{v_3} + c\underline{v_4} = \underline{0}$$

$$\left(\begin{array}{cccc|c}1 & -1 & 3 & 4 & 0\\ 2 & 1 & 1 & 0 & 0\\ 4 & 2 & 1 & 2 & 0\end{array}\right) \to \dots \to \left(\begin{array}{cccc|c}1 & -1 & 3 & 4 & 0\\ 0 & 3 & -5 & -8 & 0\\ 0 & 0 & -1 & 2 & 0\end{array}\right)$$

A

$$a - b + 3(-1) + 4c = 0$$
$$3b - 5(-1) - 8c = 0$$
$$-1(-1) + 2c = 0$$

$$a - b + 4c = 3$$
$$3b - 8c = -5$$
$$2c = -1$$

som ovenfor

Konklusjon:

Ja, $\underline{v_3}$ er lin. komb. av $\underline{v_1}, \underline{v_2}, \underline{v_4}$ på én entydig måte:

$$\underline{v_3} = 2\underline{v_1} - 3\underline{v_2} - \tfrac{1}{2}\underline{v_4}$$