	Emne	Lærebok	Oppgaver
1	Lagranges multiplikatormetode	[E] 7.7	[E] 7.7.3 - 7.7.6
2	Nødvendige betingelser	[E] 7.7	

① Lagranges multiplikatormetode

Eks: max (min) $f(x,y) = x+2y$ når $x^2+4y^2=36$

$g(x,y) = a$

D: $x^2+4y^2 = 36 \mid :36$

$\frac{x^2}{36} + \frac{4y^2}{36} = 1$

$\frac{x^2}{36} + \frac{y^2}{9} = 1$ Ellipse, Sentr. (0,0), $a=6$, $b=3$

Nivåkurver for f:

$f(x,y) = c$

$x+2y = c$

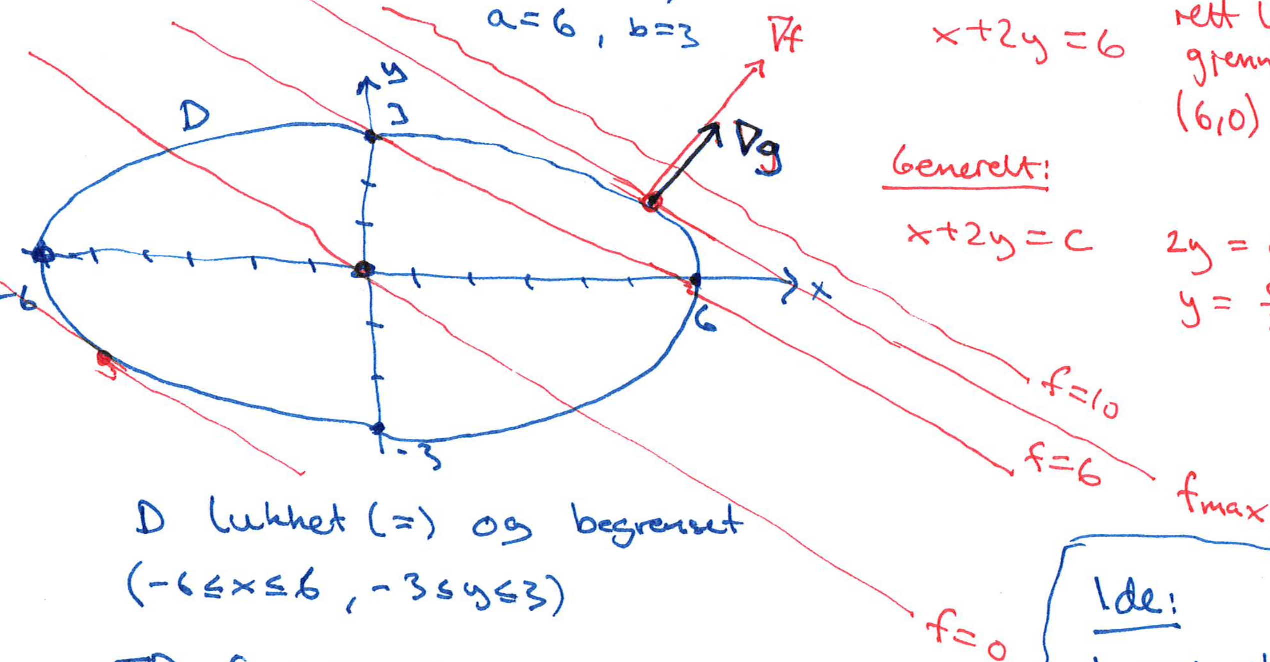
Eks: $c=6$

$x+2y=6$ rett linje gjennom (6,0) og (0,3)

Generelt:

$x+2y=c$ $\quad 2y = c-x$

$y = \frac{c}{2} - \frac{1}{2}x$



D lukket (=) og begrenset ($-6 \le x \le 6$, $-3 \le y \le 3$)

$\Rightarrow$ EVS $f = x+2y$ har max og min på D

Ide:
I maks. pkt må
$$\nabla f = \lambda \cdot \nabla g$$

Likninger:

$\nabla f = \lambda \cdot \nabla g$

$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$

$\left.\begin{array}{l} f'_x = \lambda \cdot g'_x \\ f'_y = \lambda \cdot g'_y \end{array}\right\} \rightarrow \boxed{\begin{array}{l} f'_x - \lambda \cdot g'_x = 0 \\ f'_y - \lambda \cdot g'_y = 0 \\ g(x,y) = a \end{array}}$

$+$

$g(x,y) = a$

max $f(x,y) = x + 2y$
når $g(x,y) = x^2 + 4y^2 = 36$

$\left\{\begin{array}{l} 1 - \lambda \cdot 2x = 0 \\ 2 - \lambda \cdot 8y = 0 \\ x^2 + 4y^2 = 36 \end{array}\right.$ $\left.\begin{array}{l} \\ \\ \end{array}\right\}$ FOC, C

Ide: Løsninger $(x,y;\lambda)$ av FOC+C er kandidatene for max i problemet.

λ: Lagrange-multiplikatoren

Finner kandidat punkt:

$1 - \lambda \cdot 2x = 0$
$2 - \lambda \cdot 8y = 0$
$x^2 + 4y^2 = 36$

(1) $2\lambda x = 1 \quad (\lambda \neq 0)$

$x = \frac{1}{2\lambda}$

(2) $8\lambda y = 2$

$y = \frac{2}{8\lambda} = \frac{1}{4\lambda}$

(3) $\left(\frac{1}{2\lambda}\right)^2 + 4 \cdot \left(\frac{1}{4\lambda}\right)^2 = 36$

$\frac{1}{4\lambda^2} + \frac{\cancel{4} \cdot 1}{4 \cancel{16} \lambda^2} = 36 \quad | \cdot 4\lambda^2$

$1 + 1 = 36 \cdot 4\lambda^2$

$\frac{36 \cdot 4\lambda^2}{36 \cdot 4} = \frac{\cancel{2}}{36 \cdot \cancel{4}\, 2}$

$\lambda^2 = \frac{1}{72}$ $\quad \lambda = \pm\sqrt{\frac{1}{72}} = \pm\sqrt{\frac{2}{36 \cdot 4}} = \pm\frac{\sqrt{2}}{12}$

Kandidat punkt:

$\lambda = \frac{\sqrt{2}}{12}$: $x = \frac{1}{2 \cdot \sqrt{2}/12} = \frac{1 \cdot 6}{\sqrt{2}/6 \cdot 6} = \frac{6}{\sqrt{2}}$

$y = \frac{1}{4 \cdot \sqrt{2}/12} = \frac{1 \cdot 3}{\sqrt{2}/3 \cdot 3} = \frac{3}{\sqrt{2}}$

Kandidat pkt: $(x,y;\lambda) = \left(6/\sqrt{2}, 3/\sqrt{2}; \sqrt{2}/12\right)$, $\leftarrow f = x+2y = \frac{6}{\sqrt{2}}+\frac{6}{\sqrt{2}} = \underline{\frac{12}{\sqrt{2}}}$

$\left(-6/\sqrt{2}, -3/\sqrt{2}; -\sqrt{2}/12\right) \leftarrow f = x+2y = -\frac{6}{\sqrt{2}}-\frac{6}{\sqrt{2}} = \underline{-\frac{12}{\sqrt{2}}}$

Konkl: Siden vi vet at problemet har max (fra EVS), så er

$f_{max} = \frac{12\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} = \frac{12\sqrt{2}}{2} = 6\sqrt{2} \simeq \underline{\underline{8.5}}$ i $(x,y) = \left(\frac{6}{\sqrt{2}}, 3/\sqrt{2}\right) = \left(3\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$

$\simeq \underline{\underline{(4.2, 2.1)}}$

med $\lambda = \underline{\underline{\frac{\sqrt{2}}{12}}}$

Generell metode:

Lagrange-problem: max/min $f(x,y)$ når $g(x,y) = a$

(max/min-problem med bibetingelser som er likninger)

① Skriv ned Lagrange-funksjonen $\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda(g(x,y)-a)$

② Skriv ned Lagrange-betingelsene:

$$\boxed{\begin{aligned} \mathcal{L}'_x &= f'_x - \lambda\cdot g'_x = 0 \\ \mathcal{L}'_y &= f'_y - \lambda\cdot g'_y = 0 \end{aligned}}$$ FOC (første ordens-bet.)

$\nabla f = \lambda\cdot\nabla g$, dvs nivåkurven til f tangerer D

$\mathcal{L}'_\lambda = -(g(x,y)-a) = 0$ C (bibetingelsen)

eller

$\boxed{g(x,y) = a}$ C (bibetingelsen)

$-(g(x,y)-a) = 0$

$g(x,y) - a = 0$

$g(x,y) = a$

③ Finn alle løsn. av Lagrangebetingelsene = kandidater for max/min.

Alt: $1-\lambda\cdot 2x=0$
$2-\lambda\cdot 8y=0$
$x^2+4y^2=36$

(1) $2\lambda x=1 \quad (x\neq 0)$
$\lambda=\frac{1}{2x}$

(2) $2-\lambda\cdot 8y=0$
$2-\frac{1}{2x}\cdot 8y=0$
$2=\frac{\cancel{8}^{4}y}{\cancel{2}x}=\frac{4y}{x} \quad |\cdot x$
$2x=4y$
$x=2y$

(3) $(2y)^2+4y^2=36$
$4y^2+4y^2=36$
$8y^2=36$
$y^2=\frac{\cancel{36}}{\cancel{8}}\ \frac{18}{4}$

$$y=\pm\sqrt{\frac{18}{4}}=\pm\frac{\sqrt{18}}{2}=\pm\frac{\sqrt{9}\cdot\sqrt{2}}{2}=\pm\frac{3}{2}\sqrt{2}$$

$$x=\pm 3\sqrt{2}$$

$$\lambda=\frac{1}{2x}=\pm\frac{1\cdot\sqrt{2}}{6\sqrt{2}\cdot\sqrt{2}}=\pm\frac{\sqrt{2}}{12}$$

Eks: min $f(x,y)=x^2+y^2$ når $xy=1$

Kandidatpkt:

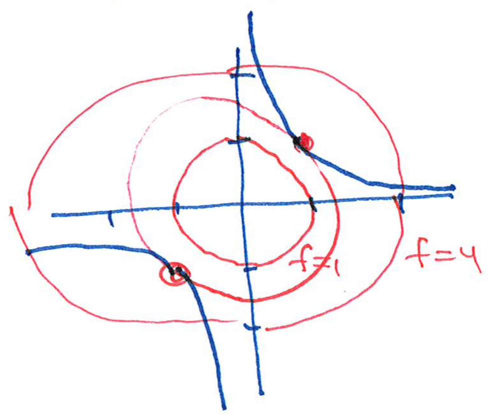
$L=x^2+y^2-\lambda(xy-1)$

$$\left.\begin{array}{l} L'_x = 2x-\lambda y=0 \\ L'_y = 2y-\lambda x=0 \end{array}\right\} \text{FOC}$$
$xy=1 \quad$ C

Forventer to kandidatpkt, begge er min-pkt, ingen max-pkt

$D: xy=1 \quad y=1/x$

D er ikke kompakt



Nivåkurver: $x^2+y^2=c$ sirkel, radius $=\sqrt{c}$

Finner kandidatpkt:

$$2x - \lambda y = 0$$
$$2y - \lambda x = 0$$
$$xy = 1$$

(1) $2x = \lambda y$

$x = \frac{\lambda}{2} \cdot y$

(2) $2y - \lambda x = 0$

$2y - \lambda\left(\frac{\lambda}{2} y\right) = 0 \;|\cdot 2$

$4y - \lambda^2 y = 0$

$y(4-\lambda^2) = 0$

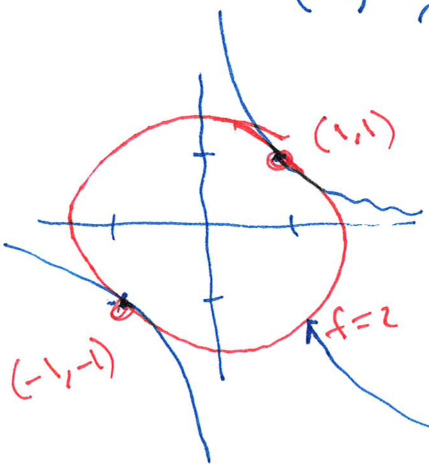
$y = 0$ eller $\lambda^2 = 4$

$y=0$	$\lambda = 2$	$\lambda = -2$
$y=0$	$x = y$	$x = -y$
(3) $xy=1$	(3) $y^2 = 1$	(3) $-y \cdot y = 1$
$x \cdot 0 = 1$	$y = \pm 1$	$y^2 = -1$
umulig	$(1,1;2)$,	umulig
	$(-1,-1;2)$	

Konkl: kandidatpkt

$(x,y;\lambda) = (1,1;2)$, $f = x^2+y^2 = 2$

$(-1,-1;2)$ $f = x^2+y^2 = 2$



$f_{min} = 2$

i $(x,y) = (1,1), (-1,-1)$

med $\lambda = 2$

siden er minste sirkel som snitter (treffer hyperbelen $xy=1$.

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