Løsning: MET11807 2022-12-19

1.

$$\left(\begin{array}{cccc|c} 2 & 9 & 5 & -9 & 7 \\ 4 & a & 10 & -18 & 22 \\ 1 & 5 & 2 & -2 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 4 & a & 10 & -18 & 22 \\ 1 & 5 & 2 & -2 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 0 & a-16 & -2 & 10 & 6 \\ 0 & 1 & -1 & 5 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & a-16 & -2 & 10 & 6 \end{array}\right)$$

Starten på Gauss-prosessen, som er felles for alle verdier av a

a) $\underline{a=22}$: setter inn ovenfor

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 6 & -2 & 10 & 6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & 4 & -20 & 12 \end{array}\right)$$

trappeform

$\underline{x} + 4y + 3z - 7w = 4$

$\underline{y} - z + 5w = -1$

$\underline{4z} - 20w = 12$

w fri

$x = -4(2) - 3(5w+3) + 7w + 4 = \underline{-8w-13}$

$y = (5w+3) - 5w - 1 = \underline{2}$

$4z = 20w + 12 \quad z = \underline{5w+3}$

Løsning: $(x,y,z,w) = \underline{(-8w-13,\ 2,\ 5w+3,\ w)}$ med w fri

uendelig mange løsninger, én frihetsgrad

b) Fortsetter Gauss-eliminasjon med generell a:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & a-16 & -2 & 10 & 6 \end{array}\right) \xrightarrow{-(a-16)} \left(\begin{array}{cccc|c} 1 & 4 & 3 & -7 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & a-18 & -5(a-18) & a-10 \end{array}\right)$$

Vi ser at $a \neq 18$ gir pivot-posisjonene markert ovenfor, mens $a=18$ gir siste rad $(0\ 0\ 0\ 0 \mid 8)$ med pivot-posisjon i siste kolonne.

Dermed har vi $\begin{cases} \text{uendelig mange løsn. for } a \neq 18 \\ \text{ingen løsninger for } a = 18 \end{cases}$

Konklusjon: Systemet er konsistent for $\underline{\underline{a \neq 18}}$.

2.

a) $$\int_0^1 7x\sqrt[3]{x}\,dx = \int_0^1 7x^{4/3}\,dx$$
$$= \left[7\cdot\frac{3}{7}\cdot x^{7/3} + C\right]_0^1 = \left[3x^2\sqrt[3]{x} + C\right]_0^1$$
$$= (3+C)-(0+C) = \underline{\underline{3}}$$

b) $$\int_0^1 \frac{9}{8-7x-x^2}\,dx = \int_0^1 \frac{-1}{x-8} + \frac{1}{x+1}\,dx = \left[-\ln|x-8| + \ln|x+1| + C\right]_0^1$$
$$= (-\ln 7 + \ln 2) - (-\ln 8 + \ln 1)$$
$$= -\ln 7 + \ln 2 + \ln(2^3)$$
$$= \underline{\underline{4\ln 2 - \ln 7}} \simeq 0.827$$

$$8-7x-x^2 = -(x-8)(x+1)$$
$$\frac{9}{8-7x-x^2} = \frac{A}{x-8} + \frac{B}{x+1} \quad \Big|\cdot(x-8)(x+1)$$
$$-9 = A(x+1) + B(x-8)$$
$$= Ax + Bx + A - 8B$$
$$-9 = (A+B)x + (A-8B)$$
$$A+B=0 \qquad B=-A \qquad B=1$$
$$A-8B=-9 \qquad A-8(-A)=9A=-9 \qquad A=-1$$

c) $$\int \frac{1}{x-\sqrt{x}}\,dx = \int \frac{1}{u^2-u}\cdot 2\sqrt{x}\,du = \int \frac{2u}{u^2-u}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}dx \end{aligned}}$$
$$x = u^2$$
$$dx = 2\sqrt{x}\,du$$

$$= \int \frac{2\cancel{u}}{\cancel{u}(u-1)}\,du = \int \frac{2}{u-1}\,du = 2\ln|u-1| + C = \underline{\underline{2\ln|\sqrt{x}-1| + C}}$$

d) $$\int_0^1 f'(x)\,dx = f(1)-f(0) \Rightarrow f(1)-f(0) = -A_1 + A_2, \text{ der}$$

A_1 = arealet mellom grafen til $f'(x)$ og x-aksen på intervallet $[0, q]$ hvor $q \simeq 0.8$ er nullpunktet til $f'(x)$ på $[0,1]$, og

A_2 = arealet mellom grafen til $f'(x)$ og x-aksen på $[q, 1]$

siden $f(x)$ er en antiderivert av $f'(x)$ per defn.

Leser av: $A_1 \simeq 8\cdot 0.2^2 = 0.32$
$A_2 \simeq 1\cdot 0.2^2 = 0.04$

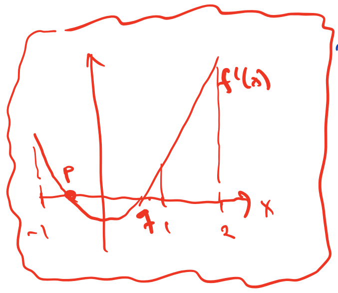
$$\Rightarrow \int_0^1 f'(x)\,dx = -A_1 + A_2 \simeq \underline{\underline{-0.28}}$$

e) max/min $f(x)$:

Kandidatpkt:

i) Randpkt $x=-1$, $x=2$

ii) Stasjonære pkt: $x=p\simeq-0.58$, $x=q\simeq 0.8$
($f'(x)=0$)



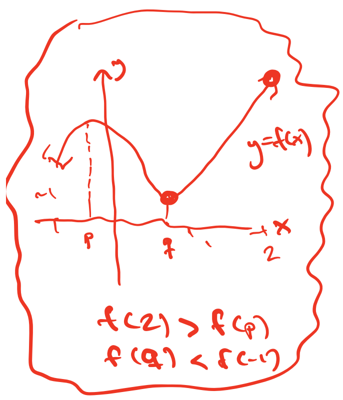
Fortegnslinje for $f'(x)$:

	-1		p		q		2
$f'(x)$		$+$	0	$-$	0	$+$	

Mulige max: $x=p$, $x=2$

$$f(2)-f(p)=\int_p^2 f'(x)\,dx = -A_{pq}+A_{q2}>0$$

der A_{pq}, A_{q2} er arealet mellom grafen til $f'(x)$ og x-aksen i $[p,q]$ og $[q,2]$, og $A_{pq}<A_{q2}$ fra figur.



Mulig min: $x=-1$, $x=q$

$$f(q)-f(-1)=\int_{-1}^{q} f'(x)\,dx = A_{-1,p}-A_{pq}<0$$

der $A_{-1,p}$ er arealet mellom grafen til $f'(x)$ og x-aksen i $[-1,p]$. Fra figur er $A_{pq}>A_{-1,p}$

Konklusjon: $x=2$ er maksimumspunktet for f
$x=q\simeq 0.8$ er minimumspunktet for f

f) Samlet nåverdi:

$$\int_0^{10} I(x)e^{-rx}\,dx=\int_0^{10} 100\cdot 1.06^x e^{-0.1x}\,dx$$

$$=100\int_0^{10}\left(e^{\ln 1.06}\right)^x\cdot e^{-0.1x}\,dx$$

$$=100\int_0^{10} e^{(\ln 1.06-0.1)x}\,dx$$

$$= \frac{100}{\ln 1.06 - 0{,}1}\left[e^{(\ln 1.06 - 0.10)x}\right]_0^{10}$$

$$= \frac{100}{\ln(1.06) - 0.1}\left(e^{10\ln 1.06 - 1} - 1\right)$$

$$= \frac{100}{\ln(1.06) - 0.10}\left(1.06^{10}/e - 1\right) \simeq \underline{\underline{817{,}6}}$$

3. $A = \begin{pmatrix} a & 2 & 3 \\ 2 & a & 3 \\ 2 & 3 & a \end{pmatrix}$

$$|A| = \begin{vmatrix} a & 2 & 3 \\ 2 & a & 3 \\ 2 & 3 & a \end{vmatrix} = a(a^2-9) - 2(2a-6) + 3(6-2a)$$
$$= a^3 - 9a - 4a + 12 + 18 - 6a = \underline{a^3 - 19a + 30}$$

utregning av $|A|$ for vilkårlig a

a) $\underline{a = 1}$:

$|A| = 12$ (setter inn)

$\Downarrow$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{12}\begin{pmatrix} -8 & 4 & 4 \\ 7 & -5 & 1 \\ 3 & 3 & -3 \end{pmatrix}^T = \underline{\underline{\frac{1}{12}\begin{pmatrix} -8 & 7 & 3 \\ 4 & -5 & 3 \\ 4 & 1 & -3 \end{pmatrix}}}$$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$C_{11} = -8$ $\quad C_{12} = 4$ $\quad C_{13} = 4$

$C_{21} = 7$ $\quad C_{22} = -5$ $\quad C_{23} = 1$

$C_{31} = 3$ $\quad C_{32} = 3$ $\quad C_{33} = -3$

b) $|A| = \underline{\underline{a^3 - 19a + 30}}$

(se ovenfor)

Ser at $|A| = a^3 - 19a + 30 = 2^3 - 19\cdot 2 + 30 = 0$ når $\underline{a = 2}$, så $a - 2$ er en faktor i uttrykket for $|A|$.

$\Downarrow$

$$|A| = a^3 - 19a + 30 = (a-2)(a^2 + 2a - 15)$$
$$= (a-2)(a-3)(a+5)$$

$$\begin{array}{l} a^3 - 19a + 30 : a - 2 = a^2 + 2a - 15 \\ \underline{a^3 - 2a^2} \\ \quad 2a^2 - 19a + 30 \\ \quad \underline{2a^2 - 4a} \\ \qquad -15a + 30 \\ \qquad \underline{-15a + 30} \\ \qquad\quad 0 \end{array}$$

$|A| = 0$ for $\underline{\underline{a = 2}}$, $\underline{\underline{a = 3}}$, $\underline{\underline{a = -5}}$

4. $f(x,y) = (4-x^2)(9-y^2) = 36 - 9x^2 - 4y^2 + x^2y^2$

a) $f'_x = -18x + 2xy^2 = 2x(y^2-9) = 0$

$f'_y = -8y + x^2 \cdot 2y = 2y(x^2-4) = 0$

$\boxed{\begin{array}{lll} x=0 & \text{eller} & y^2-9=0 \\ & & y=\pm 3 \\ y=0 & \text{eller} & x^2-4=0 \\ & & x=\pm 2 \end{array}}$

To tilfeller: a) $x=0$ og $y=0$

b) $y=\pm 3$ og $x=\pm 2$

Konklusjon: Stasjonære pkt for f er $(0,0)$, $(\pm 2, \pm 3)$,

dvs $(0,0)$, $(2,3)$, $(-2,3)$, $(2,-3)$, $(-2,-3)$

b) $f''_{xx} = -18 + 2y^2$ $\quad f''_{xy} = 2x \cdot 2y = 4xy$

$f''_{xy} = 4xy$ $\quad f''_{yy} = -8 + 2x^2$

$H(f) = \begin{pmatrix} 2y^2-18 & 4xy \\ 4xy & 2x^2-8 \end{pmatrix}$

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} -18 & 0 \\ 0 & -8 \end{pmatrix}$ $\quad tr = -26 < 0$, $\det = 144 > 0$ $\quad$ lokalt max

$(\pm 2, \pm 3)$: $H(f)(\pm 2, \pm 3) = \begin{pmatrix} 0 & \pm 24 \\ \pm 24 & 0 \end{pmatrix}$ $\quad tr = 0$, $\det = -24^2 < 0$ $\quad$ sadelpkt

Konklusjon: $(0,0)$ er et lokalt maks for f

$(\pm 2, \pm 3)$ er sadelpkt for f

c) max/min $f(x,y) = (4-x^2)(9-y^2)$

Min: Det fins ikke noe globalt min siden det ikke finnes lokale min, se b).

Max: $f(0,0) = 4 \cdot 9 = 36$ er lokalt maks.

Siden $f(4,5) = (4-16)(9-25) = (-12)(-16) = 196 > 36$, er ikke dette et globalt maks. Dermed fins ingen globale maks.

5. max $f(x,y) = 2x+4y$ når $2x^2+12x+3y^2-24y=30$

a) D: $2x^2+12x+3y^2-24y = 30$

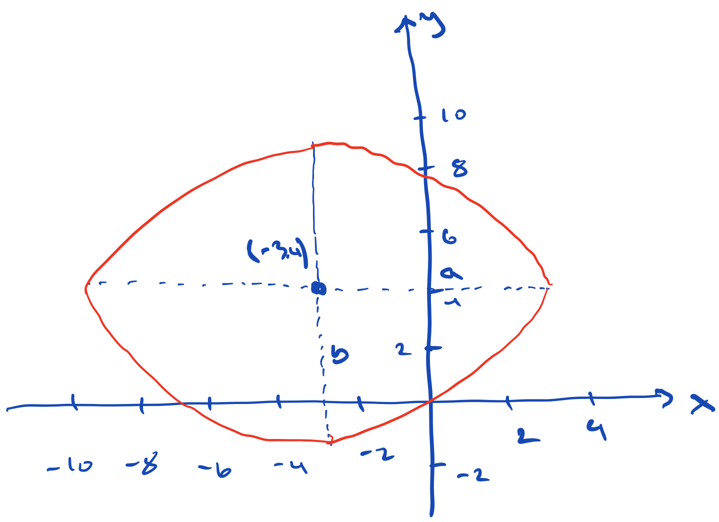
$$2(x^2+6x)+3(y^2-8y) = 30$$

$$2(x^2+6x+9)+3(y^2-8y+16) = 30+2\cdot 9+3\cdot 16 = 30+18+48 = 96$$

$$\frac{2(x+3)^2}{96} + \frac{3(y-4)^2}{96} = 1$$

$$\frac{(x+3)^2}{48} + \frac{(y-4)^2}{32} = 1$$

ellipse, senter $(-3,4)$,
halvakser $a = \sqrt{48} = 4\sqrt{3} \simeq 6.93$, $b = \sqrt{32} = 4\sqrt{2} \simeq 5.66$



D er begrenset siden
det er en ellipse
med halvakser
$a = 4\sqrt{3}$, $b = 4\sqrt{2}$,
dvs $-10 \le x \le 4$
$-2 \le y \le 10$

b) $2x^2+12x+3y^2-24y = 30$

$$4x+12+6y\cdot y' - 24y' = 0$$

$$\frac{(6y-24)y'}{6y-24} = \frac{-(4x+12)}{6y-24}$$

$$y' = -\frac{4x+12}{6y-24} = -\frac{4(x+3)}{6(y-4)} = -\frac{2}{3}\cdot\frac{x+3}{y-4}$$

$y' = -\frac{1}{2}$: $\quad -\frac{1}{2} = -\frac{2}{3}\cdot\frac{x+3}{y-4} \quad | \cdot 6(y-4)$

$$-3(y-4) = -4(x+3) \qquad \underline{y-4 = \frac{4}{3}(x+3)}$$

Setter inn i ellipselikningen:

$$2(x+3)^2 + 3(y-4)^2 = 96$$
$$2(x+3)^2 + 3\cdot(4/3)^2\cdot(x+3)^2 = 96 \quad |\cdot 3$$
$$6(x+3)^2 + 16(x+3)^2 = 96\cdot 3$$
$$22(x+3)^2 = 96\cdot 3$$

$$(x+3)^2 = \frac{96\cdot 3}{22} = \frac{48\cdot 3}{11} = \frac{12^2}{11}$$
$$x+3 = \pm\sqrt{12^2/11} = \pm 12/\sqrt{11}$$
$$\underline{x = \pm 12/\sqrt{11} - 3}$$
$$y-4 = \frac{4}{3}(x+3) = \frac{4}{3}\cdot\left(\pm 12/\sqrt{11}\right)$$
$$= \pm 16/\sqrt{11}$$
$$\underline{y = \pm\frac{16}{\sqrt{11}} + 4}$$

Vi finner to punkter:

$$x = \frac{12}{\sqrt{11}} - 3 \simeq 0.62, \quad y = \frac{16}{\sqrt{11}} + 4 \simeq 8.82$$
$$x = -\frac{12}{\sqrt{11}} - 3 \simeq -6.62, \quad y = -\frac{16}{\sqrt{11}} + 4 \simeq -0.82$$

$$\Rightarrow (x,y) = \left(\frac{12}{\sqrt{11}} - 3, \frac{16}{\sqrt{11}} - 4\right) \simeq \underline{\underline{(0.62, 8.82)}}$$

eller

$$(x,y) = \left(-\frac{12}{\sqrt{11}} - 3, -\frac{16}{\sqrt{11}} - 4\right) \simeq \underline{\underline{(-6.62, -0.82)}}$$

e) $L = 2x + 4y - \lambda(2x^2 + 12x + 3y^2 - 24y - 30)$

$$\boxed{\begin{aligned} L'_x &: 2 - \lambda(4x+12) = 0 \\ L'_y &: 4 - \lambda\cdot(6y-24) = 0 \\ & 2x^2 + 12x + 3y^2 - 24y = 3 \end{aligned}}$$

Lagrange-betingelsene

D er begrenset, så problemet har max ved ekstremverdisetn.

Ingen tillatte pkt med degenerert bibetingelse siden D er en ellipse

⇓

max = kandidatpkt med størst verdi

FOC: $\lambda = \frac{2}{4x+12} = \frac{4}{6y-24}$

$$4(4x+12) = 2(6y-24)$$
$$4\cdot 4(x+3) = 2\cdot 6(y-4)$$
$$y - 4 = \frac{16(x+3)}{12} = \frac{4}{3}(x+3)$$

← Ser at dette er samme betingelse som vi fant fra $y' = -1/2$ i b) og når vi setter inn i c får vi samme kandidatpkt som i b).

Kandidatpkt:

i) $x = \frac{12}{\sqrt{11}} - 3$, $y = \frac{16}{\sqrt{11}} + 4$, $\lambda = \frac{2}{4x+12} = \frac{1}{2(x+3)} = \frac{1}{24/\sqrt{11}} = \frac{\sqrt{11}}{24}$

$\simeq 0.62$ $\simeq 8.82$ $\simeq 0.14$

$$f = 2\left(\frac{12}{\sqrt{11}} - 3\right) + 4\left(\frac{16}{\sqrt{11}} + 4\right) = \frac{88}{\sqrt{11}} + 10 \simeq \underline{36.5}$$

ii) $x = -\frac{12}{\sqrt{11}} - 3$, $y = -\frac{16}{\sqrt{11}} + 4$, $\lambda = \frac{2}{4x+12} = \frac{1}{2(x+3)} = -\frac{\sqrt{11}}{24}$

$\simeq -6.62$ $\simeq -0.82$ $\simeq -0.14$

$$f = 2\left(-\frac{12}{\sqrt{11}} - 3\right) + 4\left(-\frac{16}{\sqrt{11}} + 4\right) = -\frac{88}{\sqrt{11}} + 10 \simeq \underline{-16.5}$$

$$\boxed{\text{Konklusjon: } f_{max} = f\left(12/\sqrt{11} - 3,\ 16/\sqrt{11} + 4\right) = 88/\sqrt{11} + 10 \simeq \underline{\underline{36.5}} \quad \text{with } \lambda = \sqrt{11}/24 \simeq 0.14}$$